LINE FIELDS ON MANIFOLDS

R. VIRK

1. **Line fields.** A line field on a smooth manifold M is a section of the projectivized tangent bundle. If such a section is only given over $M - \{x_1, \ldots, x_k\}$, for some finite number of points $x_i \in M$, then it is called a line field with singularities at the x_i. Every vector field induces a line field, but not every line field arises this way. Analogous to the Poincaré-Hopf index formula for vector fields, H. Hopf demonstrated [Hopf, Theorem 2.2]: if s is a line field with singularities $x_1, \ldots, x_k$ on a compact orientable surface Σ, then

$$\sum_i (\text{local degree of } s \text{ at } x_i) = 2\chi(\Sigma).$$

Here χ is the Euler characteristic and 'local degree'[1] is defined below.

This generalizes to higher dimensional manifolds. A treatment can be found in [CG]. The precise attribution for the general result is a bit convoluted though as several misstatements appearing in the earlier literature are corrected in [CG]. M. Grant has informed me that the first complete proof appears to be due to K. Jänich [J, §1, §2]. See [CG, §1] for a more detailed history.

I will give a simple and deliciously short proof of the general result (Corollary 3) from the 'de Rham viewpoint'. Actually, I will do a little bit more at no extra charge: an extension to projectivizations of general vector bundles is provided in Theorem 2.[2]

2. **The index.** Let $E \to M$ be an oriented $\mathbf{R}$-vector bundle of rank $2n$ over an oriented manifold M of the same dimension $2n$. Write $PE \to M$ for its projectivization and F_x for the fiber of $PE \to M$ over $x \in M$. As E is oriented, for each fiber F_x we have a preferred generator $[\sigma_x]$ of $H^{2n-1}(F_x)$ satisfying the local compatibility condition: around any point of M, there is a neighborhood U and a generator $[\sigma_U]$ of $H^{2n-1}(PE|_U)$ such that for any $x \in U$, the class $[\sigma_U]$ restricts to $[\sigma_x]$ in $H^{2n-1}(F_x)$.

Now suppose s is a section of $PE \to M$ defined over a punctured neighborhood of $x \in M$. Let D be a coordinate ball centered at x over which PE is trivial. On shrinking D if necessary, the class $[\sigma_U]$ discussed above yields an orientation of $PE|_D$. On the other hand, D inherits an orientation from M. Orient $\mathbf{RP}^{2n-1}$ so that the diffeomorphism $PE|_D \cong D \times \mathbf{RP}^{2n-1}$ preserves orientations when $D \times \mathbf{RP}^{2n-1}$ is given the product orientation. Further, let $\bar{D}$ be the closure of D. Then the orientation of D induces an orientation of the boundary $\partial\bar{D}$. The *local degree* of the section s at x is defined to be the degree, with respect to these orientations, of the composite map:

$$\partial\bar{D} \xrightarrow{s} PE|_{\bar{D}} \xrightarrow{\cong} \bar{D} \times \mathbf{RP}^{2n-1} \xrightarrow{\text{projection}} \mathbf{RP}^{2n-1}.$$

[1] Roughly, as for vector fields, the local degree (over surfaces) is the number of rotations a line makes upon traversing a small simple closed loop around the singularity. If the line field came from a vector field, then the line returns to its initial position twice for each time the vector that induced it does.

[2] Essentially the point is that the sum of the local degrees can be identified with a characteristic class for projective bundles. In the case of a projectivization, this essentially forces the class to be a multiple (easily seen to be 2) of the Euler class of the underlying vector bundle.

Proposition 1. *Let $E \to M$ be an oriented vector bundle of rank $2n$ on a compact oriented manifold M of dimension $2n$. Let $\pi\colon PE \to M$ be its projectivization. If s and t are sections of π over $M - \{x_1, \dots, x_k\}$ and $M - \{y_1, \dots, y_l\}$, respectively, then*

$$\sum_i (\text{local degree of } s \text{ at } x_i) = \sum_j (\text{local degree of } t \text{ at } y_j).$$

Proof. The fibre of $PE \to M$ is the odd dimensional projective space $\mathbf{RP}^{2n-1}$ whose De Rham cohomology vanishes except for in degrees 0 and $2n-1$. Consequently, the preferred generator of the degree $2n-1$ cohomology obtained from the orientation of E is transgressive (see [BT, Proposition 18.13]). That is, there exists a global differential form ψ on PE that restricts to the cohomology class of the preferred generator in each fibre, and is such that $d\psi = -\pi^*\tau$ for some form τ on M. Here d is the exterior derivative. For each x_i, let D_i be a coordinate ball centered at x_i. Shrinking the D_i if necessary, we may assume they do not overlap. Then

$$\int_{M - \cup_i D_i} \tau = \int_{M - \cup_i D_i} s^*\pi^*\tau = -\int_{M - \cup_i D_i} s^* d\psi = \sum_i \int_{\partial \bar{D}_i} s^*\psi$$

by Stokes' Theorem. Taking the limit as the radius of the D_i tends to 0 yields

$$\int_M \tau = \sum_i (\text{local degree of } s \text{ at } x_i).$$

Similarly for the section t. As τ is independent of the sections, the result follows. $\square$

Theorem 2. *Let $E \to M$ be an oriented vector bundle of rank $2n$ on a compact oriented manifold M of dimension $2n$. Let $PE \to M$ be its projectivization. If s is a section of PE over $M - \{x_1, \dots, x_k\}$, then*

$$\sum_i (\text{local degree of } s \text{ at } x_i) = 2 \int_M e(E),$$

where $e(E)$ is the Euler class of E.

Proof. By Proposition 1 we may replace s by a section coming from a section t of $E \to M$ with finitely many zeroes $\{y_1, \dots, y_l\}$. Such a t always exists ([BT, Proposition 11.14]), and

$$\sum_j (\text{local degree of } t \text{ at } y_j) = \int_M e(E),$$

where 'local degree of t' is the usual notion for vector bundles (see [BT, §11]). This implies the desired result. $\square$

Corollary 3. *Let s be a line field with singularities $\{x_1, \dots, x_k\}$ on a compact oriented manifold of even dimension. Then*

$$\sum_i (\text{local degree of } s \text{ at } x_i) = 2\chi(M).$$

References

[BT] R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology*, GTM **82**, Springer-Verlag (1982).

[CG] D. Crowley, M. Grant, *The Poincaré-Hopf Theorem for line fields revisited*, J. Geom. Phys. **117** (2017), 187-196.

[Hopf] H. Hopf, *Differential Geometry in the Large*, Lecture Notes in Math. **1000**, Springer-Verlag (1989).

[J] K. Jänich, *Linienfelder mit Verzweigungsdefekten*, Regensburger Mathematische Schriften **8** (1984).