SHEAVES

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1. Presheaves

1.1. Let X be a space. A *presheaf* on X is the following collection of data: a set $F(V)$ for each open subset V of X together with *restriction* maps $\mathrm{res}^V_U\colon F(V) \to F(U)$, whenever U is an open subset of V, satisfying:

(i) $\mathrm{res}^U_U = \mathrm{id}$;
(ii) $\mathrm{res}^V_U \circ \mathrm{res}^W_V$ whenever $U \subseteq V \subseteq W$.

1.2. Let F be a presheaf on a space X and let $U \subseteq X$ be an open subset. We sometimes write $\Gamma(U, F)$ instead of $F(U)$.

1.3. A *morphism* $\alpha\colon F \to G$ between two presheaves on X is a collection of maps $\alpha_U\colon F(U) \to G(U)$ for each open subset U of X such that the following diagram commutes

$$\begin{array}{ccc} F(V) & \xrightarrow{\alpha_V} & G(V) \\ {\scriptstyle \mathrm{res}^V_U}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^V_U} \\ F(U) & \xrightarrow{\alpha_U} & G(U) \end{array}$$

1.4. An element $s \in F(V)$ is called a *section* of F over V. We often write $s|_U$ instead of $\mathrm{res}^V_U(s)$ and say that $s|_U$ is the restriction of s to U.

1.5. The *stalk* F_x of F at a point $x \in X$ is

$$F_x = \varinjlim_{U \ni x} F(U),$$

where U runs through open neighborhoods of x and the morphisms in the diagram that the limit is over are the restriction maps. Thus, F_x is the disjoint union $\bigsqcup_U F(U)$ (U runs through open neighborhoods of x) modulo two sections being equivalent if they have the same restriction to some neighborhood of x. If s is a section of F over some open neighborhood of x, then the *germ* of s at x, denoted s_x, is the image of s in F_x. The germ s_x describes the behaviour of s 'arbitrarily near to x'.

1.6. Let $\alpha\colon F \to G$ be a morphism of presheaves on X. Then α induces a unique map $\alpha_x\colon F_x \to G_x$, for each $x \in X$, such that $\alpha_x(s_x) = (\alpha(s))_x$. Further, $(\mathrm{id}_F)_x = \mathrm{id}_{F_x}$. If $\beta\colon G \to H$ is another morphism of presheaves, then $(\beta\alpha)_x = \beta_x\alpha_x$.

1.7. *Remark.* Let X be a space. Let $\mathrm{Op}(X)$ be the category with objects open subsets of X and morphisms given by inclusions. A presheaf on X is equivalent to the data of a contravariant functor from $\mathrm{Op}(X)$ to the category of sets. A morphism of presheaves is equivalent to a natural transformation of such functors.

2. Sheaves

2.1. A presheaf F on X is a *sheaf* if:

(i) for any open set $U \subseteq X$, any open cover $\bigcup_{i\in I} U_i$ of U and any two sections $s, t \in F(U)$, if $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$;

(ii) for any open set $U \subseteq X$, any open cover $\bigcup_{i\in I} U_i$ of U and any family of sections $s_i \in F(U_i)$ satisfying $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i.

2.2. A presheaf satisfying (i) is called *separated.* The condition in (ii) is often called the *glueing* or *patching* condition.

2.3. **Exercise.** *Show that if F is a sheaf, then the section s obtained in (ii) above is unique.*

2.4. A morphism of sheaves is simply a morphism of presheaves. Hence, we have the category of sheaves on a space X. The next result is incredibly useful. It says that to check that a morphism of sheaves is an isomorphism, it is enough to do so at each stalk.

2.5. **Proposition.** *A morphism $\alpha\colon F \to G$ of sheaves on a space X is an isomorphism if and only if $\alpha_x\colon F_x \to G_x$ is an isomorphism for every $x \in X$.*

Proof. It is clear that if α is an isomorphism, then $\alpha_x\colon F_x \to G_x$ is an isomorphism for every $x \in X$. For the converse it suffices to show that $\alpha_U\colon F(U) \to G(U)$ is a bijection for every open subset U of X. Let $s, t \in F(U)$ and suppose $\alpha_U(s) = \alpha_U(t)$. Then, using injectivity at stalks, we infer that $s_u = t_u$ for every $u \in U$. Hence, there is an open cover $\bigcup_{i\in I} U_i$ of U such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Consequently, $s = t$ and thus, α_U is injective for every open subset U of X. Now let $r \in G(U)$. Using surjectivity at stalks, we deduce that there exists an open cover $\bigcup_{j\in J} U_j$ of U and a family of sections $s_j \in F(U_j)$ such that $\alpha_{U_j}(s_j) = t|_{U_j}$. Let $j, j' \in J$. We know that $\alpha_{U_j\cap U_{j'}}$ is injective. As $\alpha_{U_j\cap U_{j'}}(s_j|_{U_j\cap U_{j'}}) = t|_{U_j\cap U_{j'}} = \alpha_{U_j\cap U_{j'}}(s_{j'}|_{U_j\cap U_{j'}})$, it follows that the $s_j|_{U_j\cap U_{j'}} = s_{j'}|_{U_j\cap U_{j'}}$. Hence, the s_j glue to give a section $s \in F(U)$ such that $\alpha_U(s) = t$. □

2.6. **Exercise.** *Find examples of sheaves F and G such that $F_x \simeq G_x$ for every $x \in X$, but F is not isomorphic to G. Why does this not contradict the result above?*

2.7. **Exercise.** *Let F, G be sheaves on a space X. Let $\alpha, \beta\colon F \to G$ be morphisms of sheaves. Show that if $\alpha_x = \beta_x$ for all $x \in X$, then $\alpha = \beta$.*

2.8. **Example.** The sheaf of *continuous functions* $\mathcal{C}^0$ on $\mathbf{R}^n$ is defined by

$$\mathcal{C}^0(U) = \{\text{continuous functions } U \to \mathbf{R}\}.$$

Restriction maps are given by restriction of functions. There are several variations to this example: replace 'continuous' by 'smooth', 'analytic', etc., and/or replace '$\mathbf{R}^n$' by 'manifold', 'variety', 'scheme', etc.

2.9. **Example.** Define a sheaf F on $\mathbf{C}$ by

$$F(U) = \{\text{holomorphic functions } f(z) \text{ on } U \text{ satisfying } z\tfrac{df}{dz} = 1\}.$$

Restriction maps are once more given by restriction of functions. The obvious variation on this theme is to change differential equation and/or replace '$\mathbf{C}$' by 'manifold', 'variety', etc.

2.10. **Example.** Let X be a space. The *constant sheaf* on X with values in $\mathbf{Z}$, denoted $\underline{\mathbf{Z}}_X$, is defined by

$$\underline{\mathbf{Z}}_X(U) = \{\text{locally constant functions } U \to \mathbf{Z}\}.$$

The obvious variation to this theme is to replace '$\mathbf{Z}$' by any set/group. These are the sheaves that we will primarily be interested in.

2.11. **Example.** The first example and the previous one are special cases of the following more general construction. Let X and Y be spaces. Define a sheaf F on X by

$$F(U) = \{\text{continuous maps } U \to Y\}.$$

In the previous example Y was $\mathbf{Z}$ endowed with the discrete topology.

2.12. **Exercise.** *Give an example of a presheaf that is not a sheaf.*

2.13. **Exercise.** *Let F be a sheaf. Show that $F(\varnothing)$ is the one point set. (I don't particularly care about this exercise: if you like, you can take $F(\varnothing) = \{*\}$ as an additional axiom for sheaves).*

2.14. **Exercise.** *Let $f\colon X \to Y$ be a map of spaces. Define a presheaf F on Y via the assignment*

$$U \mapsto \{s\colon U \to X \mid fs = \mathrm{id}_U\},$$

where U is open in Y and restriction maps are defined in the obvious way. Show that F is a sheaf. This sheaf is called the sheaf of sections *of f.*

2.15. *Remark.* Clearly, the category of sheaves on the one point space is equivalent to the category of sets.

3. Sheafification

3.1. Given a presheaf there is a 'best possible' sheaf one can get from F: identify sections which have the same restrictions, and then add in sections for every family of sections on open covers that can be glued together. The precise construction is given below.

3.2. Let F be a presheaf on a space X. Define a presheaf $F^\#$ as follows: a section of $F^\#$ over an open subset V of X is a collection $(s_v)_{v\in V}$, where $s_v \in F_v$, such that there is an open cover $\bigcup_{i\in I} V_i$ of V and sections $s(i) \in F(V_i)$ for each $i \in I$ satisfying $s(i)_v = s_v$ for all $v \in V_i$. The restriction to an open subset U of V is given by $(s_v)_{v\in V} \mapsto (s_u)_{u\in U}$. Further, we have a morphism of presheaves $\mathrm{can}\colon F \to F^\#$ defined by $\mathrm{can}(s) = (s_v)_{v\in V}$, where $s \in F(V)$.

3.3. **Proposition.** *The presheaf $F^\#$ is a sheaf. Further:*

(i) $\mathrm{can}_x\colon F_x \xrightarrow{\sim} F^\#_x$ *is an isomorphism for all $x \in X$;*
(ii) *if G is a sheaf and $\alpha\colon F \to G$ is a morphism of presheaves, then there is a unique morphism of sheaves $\beta\colon F^\# \to G$ such that $\beta \circ \mathrm{can} = \alpha$.*

3.4. **Exercise.** *Supply a proof.*

3.5. The sheaf $F^{\#}$ is called the *sheafification* of F. In view of the proposition above it is clear that if F is a sheaf then $\mathrm{can}\colon F \xrightarrow{\sim} F^{\#}$ is an isomorphism.

3.6. *Remark.* Let X be a space and let $\mathrm{Sh}(X)$ and $\mathrm{PreSh}(X)$ denote the category of sheaves and presheaves on X respectively. Then sheafification gives a functor ${}^{\#}\colon \mathrm{PreSh}(X) \to \mathrm{Sh}(X)$. We also have a forgetful functor $\mathrm{For}\colon \mathrm{Sh}(X) \to \mathrm{PreSh}(X)$. The result above may be reformulated as the statement: sheafification is left adjoint to the forgetful functor.

3.7. **Exercise.** *Show that the sheafification of a presheaf is unique up to unique isomorphism.*

4. Abelian sheaves

4.1. An *abelian sheaf* (or sheaf of abelian groups) is a sheaf F on a space X, such that, for each open subset V of X, the set of sections $F(V)$ is an abelian group and the restriction $F(V) \to F(U)$ is a group homomorphism for each open subset U of V. If one replaces 'sheaf' with 'presheaf', one also has the concept of an abelian presheaf. A morphism $\alpha\colon F \to G$ of abelian presheaves is a morphism of presheaves such that $\alpha_U\colon F(U) \to G(U)$ is a group homomorphism for every open subset U of X. A morphism of abelian sheaves is simply a morphism of abelian presheaves.

4.2. Replacing 'abelian group' with 'k-vector space' (for some fixed field k) and replacing 'group homomorphism' by 'k-linear map' we obtain the notion of a sheaf of vector spaces. From now on, unless explicitly stated otherwise, 'sheaf' = 'abelian sheaf'.

4.3. **Exercise.** *Verify that all the constructions of the previous sections go through for abelian (pre-)sheaves. Namely, stalks admit a natural structure of abelian group, all the morphisms constructed are morphisms of abelian groups, sheafification of an abelian presheaf gives an abelian sheaf, the sheafification of an abelian presheaf is an abelian sheaf, etc. Do the same for sheaves of vector spaces.*

4.4. **Warning.** It is tempting to consider sheaves of objects in an abelian category. However, these do not have all the nice properties of sheaves of abelian groups/vector spaces. Several of the issues that arise can be traced to the behaviour of limits/colimits in these categories. For instance, taking filtrant colimits of abelian groups is an exact functor. This statement is *not generally true* for arbitrary abelian categories. So some care needs to exercised in the general situation.

5. Support of a section

5.1. Let F be a sheaf on a space X. Let s be a section of F over an open set $U \subseteq X$. The *support* of s, denoted $\mathrm{supp}\, s$, is

$$\mathrm{supp}\, s = \{x \in U \mid s_x \neq 0\}.$$

5.2. **Exercise.** *Show that* $\mathrm{supp}\, s$ *is the complement in* U *of the union of open subsets* $V \subseteq U$ *such that* $s|_V = 0$. *In particular,* $\mathrm{supp}\, s$ *is closed in* U.

6. Direct sums and products of sheaves

6.1. Let $\{F_i\}_{i\in I}$ be a family of sheaves on a space X. The direct sum (= coproduct) $\bigoplus_i F_i$, and the product $\prod_i F_i$ are defined in the obvious way. Namely, if $U \subseteq X$ is open, then

$$\left(\bigoplus_i F_i\right)(U) = \bigoplus_i F_i(U) \quad \text{and} \quad \left(\prod_i F_i\right)(U) = \prod_i F_i(U),$$

with restriction maps defined in the obvious way.

7. Exact sequences of sheaves

7.1. Let $\alpha\colon F \to G$ be a morphism of presheaves on a space X. Define a presheaf $\ker(\alpha)$ by $\ker(\alpha)(U) = \ker(\alpha_U\colon F(U) \to G(U))$, restriction maps are those induced by F. If F and G are sheaves, then $\ker(\alpha)$ is also a sheaf.

7.2. Define a presheaf $\text{pre}-\text{im}(\alpha)$ by $\text{pre}-\text{im}(\alpha)(U) = \text{im}(\alpha_U\colon F(U) \to G(U))$, restriction maps are those induced by G. Even if F and G are sheaves, $\text{pre}-\text{im}(\alpha)$ is *not* in general a sheaf. If F and G are sheaves, we write $\text{im}(\alpha)$ for the smallest subsheaf of G containing $\text{pre}-\text{im}(\alpha)$. More precisely, a section of $\text{im}(\alpha)$ over an open subset U of X is a section $s \in G(U)$ such that there is an open cover $\bigcup_{i\in I} U_i$ of U with $s|_{U_i} \in \alpha_U(U_i)$ for all $i \in I$.

7.3. **Exercise.** *Show that* $\text{im}(\alpha) \simeq (\text{pre}-\text{im}(\alpha))^{\#}$.

7.4. **Exercise.** *Give an example of a morphism of sheaves* $\alpha\colon F \to G$ *on some space* X *such that* $\text{pre}-\text{im}(\alpha)$ *is not a sheaf.* Aside: *The easiest example I could come up with involved sheaves of continuous maps on* S^1. *I would be very interested if you can come up with a simpler 'naturally occuring' example.*

7.5. A sequence of morphisms

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

of sheaves on a space X is called *exact* if $\ker(\beta) = \text{im}(\alpha)$. The next result says that no matter how complicated our sheaves are, exactness is a local issue.

7.6. **Proposition.** *A sequence of morphisms*

$$F \to G \to H$$

of sheaves on a space X *is exact if and only if*

$$F_x \to G_x \to H_x$$

is exact for every $x \in X$.

7.7. **Exercise.** *Prove this.*

7.8. A sequence of morphisms

$$\cdots \to F^i \to F^{i+1} \to \cdots$$

of sheaves on a space X is exact if $F^{i-1} \to F^i \to F^{i+1}$ is exact for each i. An exact sequence of the form $0 \to F \to G \to H \to 0$ is also called a short exact sequence.

7.9. **Exercise.** *Let* $\alpha\colon F \to G$ *be a morphism of sheaves on a space* X. *Appropriately formulate the notion of the sheaf cokernel* $\text{coker}(\alpha)$.

7.10. Let $0 \to F \to G \to H \to 0$ be a sequence of morphisms of sheaves on a space X. If $0 \to F(U) \to G(U) \to H(U) \to 0$ is exact for every open subset U of X, then $0 \to F \to G \to H \to 0$ is an exact sequence. However, the converse is not true.

7.11. **Proposition.** *Let*

$$0 \to F \to G \to H \to 0$$

be an exact sequence of sheaves on a space X. *Then*

$$0 \to F(U) \to G(U) \to H(U)$$

is exact for every open subset U *of* X.

7.12. **Exercise.** *Prove this. Give an example of an exact sequence of sheaves* $0 \to F \to G \to H \to 0$ *on a space* X *such that* $0 \to F(U) \to G(U) \to H(U) \to 0$ *is not exact for some open subset* U *of* X.

7.13. **Example.** Let X be a space. Let $\mathbf{Z}(1) = \{(2\pi\sqrt{-1})n \mid n \in \mathbf{Z}\}$. Then $\mathbf{Z}(1)$ is a subgroup of $\mathbf{C}$. Let $\mathbf{C}^* = \mathbf{C} - \{0\}$ viewed as an abelian group under multiplication. The inclusion $\mathbf{Z}(1) \hookrightarrow \mathbf{C}$ and the map $\mathbf{C} \to \mathbf{C}^*$, $z \mapsto e^z$, induce an exact sequence of sheaves

$$0 \to \underline{\mathbf{Z}}_X(1) \to \underline{\mathbf{C}}_X \xrightarrow{\exp} \underline{\mathbf{C}}^*_X \to 1,$$

where $\underline{\mathbf{Z}}_X(1)$, $\underline{\mathbf{C}}_X$, $\underline{\mathbf{C}}^*_X$ are the constant sheaves with values in $\mathbf{Z}(1)$, $\mathbf{C}$ and $\mathbf{C}^*$ respectively (see Example 2.10). If X is a complex manifold, then replacing $\underline{\mathbf{C}}_X$ with the structure sheaf $\mathcal{O}_X$ of holomorphic functions on X and replacing $\underline{\mathbf{C}}^*_X$ by the sheaf $\mathcal{O}^*_X$ of nowhere vanishing holomorphic functions, one obtains the classical *exponential sheaf sequence.*

7.14. **Exercise.** *Let $\mathcal{O}_{\mathbf{C}}$ denote the sheaf of complex valued holomorphic functions on $\mathbf{C}$. Let $\frac{d}{dz}\colon \mathcal{O}_{\mathbf{C}} \to \mathcal{O}_{\mathbf{C}}$ be the derivative in the coordinate z. Show that the sequence*

$$0 \to \underline{\mathbf{C}}_{\mathbf{C}} \xrightarrow{i} \mathcal{O}_{\mathbf{C}} \xrightarrow{\frac{d}{dz}} \mathcal{O}_{\mathbf{C}} \to 0$$

is exact. Here i is the evident inclusion.

7.15. We say that a morphism of sheaves is injective (resp. surjective) if the induced map on stalks is injective (resp. surjective).

7.16. **Exercise.** *Let $\alpha\colon F \to G$ be an injective morphism of sheaves on a space X. Is it true that $\alpha_U\colon F(U) \to G(U)$ is injective for open subsets U of X? What about if we replace 'injective' with 'surjective'?*

8. Additional exercises

8.1. Here are some exercises motivated by some questions that have been asked in class.

8.2. **Exercise.** *Determine whether the following presheaves on $\mathbf{C}$ are sheaves (restriction maps are the obvious ones):*

(i) $U \mapsto \{$*bounded continuous functions* $U \to \mathbf{C}\}$.
(ii) $U \mapsto \{$*continuous functions* $f\colon U \to \mathbf{C}$ *such that* $f(z)^2 = z\}$.
(iii) $U \mapsto \{$*continuous functions* $f\colon U \to \mathbf{C}$ *such that there exists a continuous function* $g\colon U \to \mathbf{C}$ *satisfying* $g^2 = f\}$.

8.3. **Exercise.** *What are the limits and colimits of the following diagrams in the category of abelian groups:*

(i) $\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \cdots$.
(ii) $\cdots \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \mathbf{Z}$.
(iii) $\cdots \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \cdots$.

8.4. **Exercise.** *Meditate on the following: let X be a space. Let $\mathrm{Op}(X)$ be the category with objects open sets of X, and morphisms given by inclusion maps. A presheaf (of sets) is a contravariant functor $F\colon \mathrm{Op}(X) \to \mathbf{Set}$. As before, if $U \subseteq V$ is the inclusion of open sets, we will call the corresponding map $F(V) \to F(U)$ the restriction map. A presheaf is a sheaf if for every $U \in \mathrm{Op}(X)$ and for every open cover $\bigcup_{i\in I} U_i$ of U the limit of the diagram*

$$\textstyle\prod_{i\in I} F(U_i) \overset{\mathrm{res}_1}{\underset{\mathrm{res}_2}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(U_i \cap U_j)$$

is $F(U)$. Here res_1 is the map induced by the family of restriction maps $F(U_i) \to F(U_i \cap U_j)$, and res_2 is the map induced by the family of restriction maps $F(U_j) \to F(U_i \cap U_j)$.

Alternatively, a presheaf F is a sheaf if, for every $U \in \mathrm{Op}(X)$, and every open cover $\bigcup_{i\in I} U_i$ of U that is stable under finite intersections, the morphism

$$F(U) \to \varprojlim_{i\in I} F(U_i)$$

is an isomorphism (the morphisms in the limit diagram are the restriction maps).

8.5. **Exercise.** *Let X be a space and let F be the presheaf on X defined by $U \mapsto \mathbf{Z}$, restriction maps are the identity map. Let G be a sheaf on X. Make sense of the following statement: to define a morphism $\underline{\mathbf{Z}}_X \to G$ it is sufficient to define a morphism $F \to G$, further every morphism $\underline{\mathbf{Z}}_X \to G$ arises from a morphism $F \to G$.*

8.6. **Exercise.** *Consider the category of commutative rings with 1 (morphisms are required to send 1 to 1). What is the coproduct of two objects in this category? Now consider the category of commutative rings with 1 that have no nilpotent elements (morphisms are again required to send 1 to 1). What is the coproduct of two objects in this category?*

8.7. **Exercise.** *What is the coproduct in the category of sets? What is the coproduct in the category of abelian groups? Does the coproduct on abelian groups agree with that on the underlying sets?*

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