Satake isomorphisms and affine Beilinson-Bernstein localization

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$G \supset B \supset T$

F = finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$

$\mathcal{O} = \mathbb{Z}_p$ or $\mathbb{F}_q[[t]]$

t = uniformizer

Local Langlands philosophy

$$\mathrm{Gal}(\bar{F}/F) \longrightarrow G^\vee_{\mathbb{C}}$$

$$\left\{ \begin{array}{l} \text{irred. reps} \\ \text{of } G(F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{fin. dim reps} \\ \text{of } \mathrm{Gal}(\bar{F}/F) \end{array} \right\}$$

$$\bigcup$$

$$\left\{ \begin{array}{l} \text{unramified reps} \\ \text{i.e. } \exists\, G(\mathcal{O})\text{-fixed} \\ \text{vector} \end{array} \right\} \qquad \left\{ \begin{array}{l} \text{unramified parameters} \\ \text{i.e. } \mathrm{Gal}(\bar{F}/F) \longrightarrow GL(V) \end{array} \right\}$$

$$\begin{tikzcd} \mathrm{Gal}(\bar{F}/F) \arrow[r] \arrow[d] & GL(V) \\ \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \arrow[ur, dotted] & \end{tikzcd}$$

$\mathrm{ind}_{B(F)}^{G(F)} \chi \quad \leftarrow$ almost always irred.

$\chi|_{B(\mathcal{O})} = 1$

Satake isomorphism

$$\mathrm{Fun}(G(\mathcal{O})\backslash G(F)/G(\mathcal{O})) \xrightarrow{\sim} \mathbb{C}[T^\vee/W] \simeq K_0(\mathrm{Rep}\, G^\vee)$$

$$\circledast \quad \| \quad \mathcal{H}$$

$W = \mathrm{ind}_{G(\mathcal{O})}^{G(F)} 1 = \mathrm{fun}(G(F)/G(\mathcal{O}))$

$\mathcal{H} \curvearrowright W$; maximal ideals in $\mathcal{H}$

$\updownarrow$

points in $T^\vee/W \longleftrightarrow$ characters of $B(F)$ trivial on $B(\mathcal{O})$

$\mathfrak{m} \subseteq \mathcal{H} \rightsquigarrow G(F)$-module $W/\mathfrak{m}W$

for almost all $\mathfrak{m}$, $W/\mathfrak{m}W$ is an irred. unramified rep.

$$W \hookleftarrow \mathcal{H} \longleftrightarrow \left\{ \begin{array}{c} \text{unramified} \\ \text{parameters} \end{array} \right\}$$

<u>Ramified Satake isomorphism</u>

$\bar{\mu} : T(\mathcal{O}) \longrightarrow \mathbb{C}^*$

Assume $\mathrm{stab}(\mu)$ in the Weyl group is parabolic. Let L be the corresponding Levi.

<u>Thm</u> $\exists$ a subgroup $K \subseteq G(\mathcal{O})$, $K \supseteq T(\mathcal{O})$ and a character $\mu : K \longrightarrow \mathbb{C}^*$ extending $\bar{\mu}$, s.t.

(Howe, Bushnell-Kutzko, Roche, K.-Schedler)

$$\mathcal{H}(G(F), K, \mu) \xrightarrow{\sim} \mathbb{C}[T^\vee/W_{\bar{\mu}}]$$

$f : G(F) \longrightarrow \mathbb{C}$

$f(kgk') = \mu(k) f(g) \mu(k')$

Geometry

$F = \mathbb{F}_q((t))$

$\mathrm{Gal}(\overline{F}/F)$ - rep
$= \pi_1(D^\times_{\mathbb{F}_q})$ - rep

Geometric analogue:
meromorphic $G^\vee$ local systems on $D^\times_{\mathbb{C}}$

What are geometric analogue of reps. on the other side? Mysterious!

Consider D-modules on $\underbrace{G((t))/B((t))}_{\text{but this is a hard space!}}$

↑ so this doesn't quite work

Geometric Satake

$$P(G[[t]] \backslash G((t)) / G[[t]]) \simeq \mathrm{Rep}(G^\vee)$$

Thm (Gaitsgory) $P(G[[t]] \backslash G((t)) / G[[t]]) \hookrightarrow P(G((t))/G[[t]])$

Thm (K.-Schedler) $\bar{\mu}$ regular, then

① $P^{\bar{\mu}\times\bar{\mu}}(K \backslash G((t))/K) \simeq \mathrm{Rep}(T^\vee)$

② $P^{\bar{\mu}\times\bar{\mu}}(K \backslash G((t))/K) \hookrightarrow P^{\bar{\mu}}(G((t))/K)$

$$\mathrm{Fun}\left(G(\mathbb{F}_q)/B(\mathbb{F}_q)\right) \rightsquigarrow \text{D-modules on } G/B$$

$$\updownarrow \text{localization}$$

$$\rightsquigarrow \mathfrak{g}\text{-modules}$$

<u>Reps. of affine Kac-Moody algebras</u>

$\cancel{1 \to \mathbb{C} \to \mathfrak{g}((t))}$

$$1 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 1$$

$\uparrow$ affine Kac-Moody algebra

central extensions classified by

$H^2(\mathfrak{g}((t)), \mathbb{C}) = \mathbb{C}$

Given $\kappa \in \mathbb{C}$, $\exists$ $\hat{\mathfrak{g}}_\kappa$ affine Kac-Moody at level κ. OK we want 'critical level' reps. + 'smooth'.

$\rightsquigarrow$ $\tilde{U}_c(\hat{\mathfrak{g}})$ - mod

<u>Feigin - Frenkel</u> $\quad Z(\hat{U}_c(\hat{\mathfrak{g}})) \simeq \mathbb{C}[G^\vee\text{-opers on } D^\times_{\mathbb{C}}]$

$G^\vee$-oper is a $G^\vee$ local system on $D^\times_{\mathbb{C}}$ + additional data.

$$\begin{array}{ccc} G^\vee\text{-opers} & & \chi = \chi_{z\sigma} \\ \downarrow \text{forget} & & \downarrow \\ G^\vee\text{- local systems on } D^\times_{\mathbb{C}} & \ni & \sigma \end{array}$$

Frenkel - Gaitsgory proposal

$$\mathcal{C}_\sigma := \left(\hat{\mathfrak{g}}_\kappa\text{-mod}\right)_{\chi_\sigma}$$

$$\mathcal{H} \curvearrowright W \longleftrightarrow \left\{\begin{array}{c}\text{unramified}\\ \text{parameters}\end{array}\right\}$$

$\rightsquigarrow$ $\sigma = \sigma_0$ trivial $G^\vee$-local system on $D^\times_{\mathbb{C}}$

$$\begin{tikzcd}
\text{D-mod}(G((t))/G[[t]]) \arrow[dr, "\Gamma \text{ (exact)}"] & \\
\text{D-mod}(G[[t]]\backslash G((t))/G[[t]]) \arrow[u] & (\hat{\mathfrak{g}}_\kappa\text{-mod})_{\sigma_0} \\
\text{Rep}(G^\vee) \arrow[u, equal] & \\
& D^{\text{Hecke}}\text{-mod}(G((t))/G[[t]]) \arrow[uu, "\Gamma \text{ (exact + fully faithful)}"']
\end{tikzcd}$$

conjecturally an equivalence.

$$\begin{tikzcd}
\text{D-mod}(G((t))/G[[t]]) \arrow[r, "\Gamma"] & (U_\kappa(\hat{\mathfrak{g}})\text{-mod})_{\sigma_0} \\
\text{Rep}(G^\vee) \arrow[u] & \\
\rightsquigarrow D^{\text{Hecke}}\text{-mod}(G((t))/G[[t]]) \arrow[uur, "\Gamma"'] &
\end{tikzcd}$$