E. van Erp — Index theory for Elliptic operators

Fredholm (1903)

$\lambda \in \mathbb{C} - \{0\}$

Thm (1) $\lambda u(x) - \int_a^b K(x,y)\, u(y)\, dy = 0$

(2) —— " —— $= f(x)$

K - cont. on $[a,b] \times [a,b]$

Either (1) has no non-trivial sols. $u \neq 0$

or (2) has solutions for 'all' f.

Reformulated: $H = L^2([a,b])$

$K \in \mathcal{K}(H) \leftarrow$ compact operator

Thm $\dim \ker(\lambda - K) > 0$

or $\dim \operatorname{coker}(\lambda - K) = 0$.

Def Operator $T \in \mathcal{L}(H) \leftarrow$ bounded linear operator

is Fredholm if:

1) $\exists S \in \mathcal{L}(H)$ s.t. $TS - 1,\ ST - 1 \in \mathcal{K}(H)$

$\Leftrightarrow$ 2) $\dim \ker T < \infty$, $\dim \operatorname{coker} T < \infty$.

Fredholm index T = $\dim \ker T - \dim \operatorname{coker} T$

Fact index T_0 = index T_1 iff

$\exists$ a homotopy T_t, $t \in [0,1]$ of Fredholm operators $T_0 \frown T_1$

Fredholm's theorem

$$\text{index } \lambda \mathbb{1} - K = \text{index } \lambda \cdot \mathbb{1} = 0.$$

Index formulas

Analytic formulas: $\text{index } T = \operatorname{Tr}(1-ST) - \operatorname{Tr}(1-TS)$

$$= \operatorname{Tr}(e^{-T^*T}) - \operatorname{Tr}(e^{-TT^*})$$

$$= \text{etc.}$$

want: 'topological' formulas

Example $S^{2n+1} \subset \mathbb{C}^{n+1}$ unit sphere

Hardy space $\underset{H}{H^2(S^{2n+1})} \subset L^2(S^{2n+1})$

completion of $\{f$ cont. on $\bar{B}$ and holom. on $B^\circ\}$

Szegö projection: $L^2 \to H^2$

Def Toeplitz operator $T_f \in \mathcal{L}(H^2)$ is an operator of form

$$T_f = SM_f : H^2 \xrightarrow[M_f]{} L^2 \xrightarrow[S]{} H^2$$

where

$$f : S^{2n+1} \longrightarrow \mathbb{C} \quad \text{continuous fns.}$$

$\mathcal{T}$ = Toeplitz alg = alg. gend. by $\{T_f\}$ (norm closed)

Lemma $T_f T_g = T_{fg}$ + compact

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow[\sigma]{T_f \mapsto f} C(S^{2n+1}) \to 0$$

T_f is Fredholm $\iff$ $f : S^{2n+1} \longrightarrow \mathbb{C} \setminus \{0\}$

simplest case $n=0$ $f : S^1 \longrightarrow \mathbb{C}\setminus\{0\}$

Thm on S^1 Index $T_f = -\text{winding}(f)$.

$$= -\frac{1}{2\pi i} \int_{S^1} f^{-1} df \qquad \text{(if } f \text{ is smooth)}$$

proof Explicitly compute for

$$f : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}\setminus\{0\}$$
$$e^{i\theta} \longmapsto e^{ik\theta}$$

~~that~~ T_f has index $-k$.

On S^{2n+1}: systems of operators

$$H = H^2(S^{2n+1}) \oplus \cdots \oplus H^2 = H^2 \otimes \mathbb{C}^r$$

$$f : S^{2n+1} \longrightarrow GL(r, \mathbb{C})$$

$$T_f \circlearrowleft H^2 \otimes \mathbb{C}^r$$

Thm

$$\text{Index } T_f = c_n \int_{S^{2n+1}} \mathrm{Tr}(f^{-1} df)^{2n+1} \qquad f \in \mathcal{C}^\infty$$

Remarks - $f \in \pi_{2n+1}(GL(r,\mathbb{C})) \longrightarrow \pi_{2n+1}(GL(\infty,\mathbb{C}))$
$\cong \mathbb{Z}$ Bott periodicity.

- can rephrase the formula:

$$M = S^{2n+1} \times S^1$$

$E_f \longmapsto M$ vector bundle w/ fibre $\mathbb{C}^r$

$$[E_f] \in K^0(M) \otimes \mathbb{Q}$$

$$\simeq \Big\downarrow \text{ chern character}$$

$$\bigoplus_j H^{2j}(M)$$

fredholm operator $\rightsquigarrow$ symbol $\rightsquigarrow$ cohomology class

__Thm__ $\quad \text{Index } T_f = \int_M ch(E_f)$

__Atiyah - Singer__ (1961) Topological index formula for elliptic differential operators.

Recently - extended to non-elliptic differential operators

__Example__ $\quad S^{2n+1} \subset \mathbb{C}^{n+1}$

$H = \mathbb{R}$ - codim 1 sub-bundle TS^{2n+1}

closed under mult. w/ $\sqrt{-1}$

choose an orthonormal frame locally on S^{2n+1}

$$\underbrace{W_1, \ldots, W_{2n}}_{\text{span } H}, \; T$$

T ← Reeb field

$$\Delta_H = -\sum_{j=1}^{2n} W_j^2 \qquad \text{(sub laplacian)}$$

Non-elliptic operator:

$$P_f = \Delta_H + ifT = \Delta_H \otimes \mathbb{1}_r + iT \otimes f$$

acts on $C^\infty(S^{2n+1}, \mathbb{C}^r)$

$$f : S^{2n+1} \longrightarrow M(r, \mathbb{C})$$

Thm P_f is Fredholm iff $\forall x \in S^{2n+1}$

$f(x) - k \mathbb{1}_r \in GL(r, \mathbb{C})$, smooth

for all $k = \pm n, \pm(n+2), \pm(n+4), \ldots$

Thm [E v E, ~~Pal~~ Paul Baum]

$$\text{Index } P_f = \sum_{\substack{k = n+2j \\ j = 0,1,2,\ldots}} \int_{M = S^{2n+1} \times S} ch(E_f^k \otimes \mathrm{Sym}^j H^{1,0})$$

$$+ \sum_{\substack{k = -(n+2j) \\ j = 0,1,2,\ldots}} \int_{\overline{M}} ch(E_f^k \otimes \mathrm{Sym}^j \overline{H}^{1,0})$$