SPRINGER'S HOMOTOPY ARGUMENT

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In what follows 'variety' = 'scheme' = 'separated scheme of finite type over $\mathbf{C}$'. A point will always mean a closed point. By sheaf we mean a constructible sheaf (in the analytic topology) of vector spaces over some algebraically closed field (which may as well be assumed to be $\mathbf{C}$). We write $\mathbf{R}f_*$ for the right derived functor of the direct image functor associated to a continuous map f and we write $\mathbf{R}^i f_*$ for the i^{th}-cohomology of $\mathbf{R}f_*$.

The terms 'functorial', 'natural' and 'canonical' will be used as synonyms for 'a morphism of functors'. For a functor F, we write $\mathbb{1}_F$ for the identity endomorphism of F.

1.1. Let V be a variety, $\alpha\colon \mathbf{C}^* \times V \to V$ a $\mathbf{C}^*$-action. Let $\pi\colon \mathbf{C}^* \times V \to V$ be the projection map. A $\mathbf{C}^*$-*equivariant* sheaf on V is a tuple $(\mathcal{A}, \sigma)$, where $\mathcal{A}$ is a sheaf on V and σ is an isomorphism $\alpha^*\mathcal{A} \xrightarrow{\sim} \pi^*\mathcal{A}$. The isomorphism σ is required to satisfy a cocycle condition whose details we omit, since this condition is not relevant to the discussion at hand (see [Sc, §3.1] for details). Further, we will simply say that $\mathcal{A}$ is a $\mathbf{C}^*$-equivariant sheaf and omit the isomorphism σ from our notation.

1.2. Let V be a variety endowed with a $\mathbf{C}^*$-action $\alpha\colon \mathbf{C}^* \times V \to V$. Additionally, assume that the $\mathbf{C}^*$-action *contracts* V to some point $x_0 \in V$. That is, the action α extends to a morphism $\mu\colon \mathbf{C} \times V \to V$ such that $\mu(0, x) = x_0$ for all $x \in V$.

Let $\delta\colon \{x_0\} \hookrightarrow V$ and $a\colon V \twoheadrightarrow \{x_0\}$ be the inclusion and projection maps respectively.

Lemma. *Let $\mathcal{A}$ be a $\mathbf{C}^*$-equivariant sheaf on V. Assume that $\delta^*\mathcal{A} = 0$ (i.e., the stalk at x_0 is trivial). Then $\mathbf{R}a_*\mathcal{A} = 0$.*

Proof. Let $\pi_1\colon \mathbf{C} \times V \to \mathbf{C}$ and $\pi_2\colon \mathbf{C} \times V \to V$ be the projection maps. Then we have a cartesian square

$$\begin{tikzcd} \mathbf{C} \times V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & V \arrow[d, "a"] \\ \mathbf{C} \arrow[r] & \{x_0\} \end{tikzcd}$$

So, by smooth base change, it suffices to show that $\mathbf{R}\pi_{1*}\pi_2^*\mathcal{A} = 0$. Note that smooth base change already implies that $\mathbf{R}^i\pi_{1*}\pi_2^*\mathcal{A}$ is a constant sheaf on $\mathbf{C}$ for each i.

Define $\tau\colon \mathbf{C} \times V \to \mathbf{C} \times V$ by $(t, x) \mapsto (t, \mu(t, x))$. Then τ is a morphism over $\mathbf{C}$, i.e., $\pi_1\tau = \pi_1$. As $\mathcal{A}$ is $\mathbf{C}^*$-equivariant, the restriction of $\pi_2^*\mathcal{A}$ to $\mathbf{C}^* \times V$ is isomorphic to $\alpha^*\mathcal{A}$. Now $\alpha^*\mathcal{A}$ is the restriction of $(\pi_2\tau)^*\mathcal{A} = \tau^*\pi_2^*\mathcal{A}$ to $\mathbf{C}^* \times V$. Thus, the restrictions to $\mathbf{C}^* \times V$ of $\pi_2^*\mathcal{A}$ and $\tau^*\pi_2^*\mathcal{A}$ are isomorphic. The restriction of $\tau^*\pi_2^*\mathcal{A}$ to $\{0\} \times V$ is zero, since the stalk $\mathcal{A}_{x_0}$ is zero. Thus, the sheaf $\tau^*\pi_2^*\mathcal{A}$ is the extension by zero of its restriction to $\mathbf{C}^* \times V$. In formulas: let $j\colon \mathbf{C}^* \times V \hookrightarrow \mathbf{C} \times V$ be the inclusion map, then

$$\tau^*\pi_2^*\mathcal{A} \simeq j_!j^*\tau^*\pi_2^*\mathcal{A} \simeq j_!j^*\pi_2^*\mathcal{A}.$$

As j is an open inclusion, $j_!$ is left adjoint to j^*. In particular, we have a canonical map $\varepsilon'\colon j_!j^* \to \mathrm{id}$. Similarly, we have a canonical map $\eta'\colon \mathrm{id} \to \mathbf{R}\tau_*\tau^*$.

Define $\gamma\colon \mathbf{R}\pi_{1*}\pi_2^*\mathcal{A} \to \mathbf{R}\pi_{1*}\pi_2^*\mathcal{A}$ to be the composition

$$\mathbf{R}\pi_{1*}\pi_2^*\mathcal{A} \xrightarrow{\mathbb{1}_{\mathbf{R}\pi_{1*}}\eta'\mathbb{1}_{\pi_2^*}} \mathbf{R}\pi_{1*}\mathbf{R}\tau_*\tau^*\pi_2^*\mathcal{A} = \mathbf{R}(\pi_1\tau)_*\tau^*\pi_2^*\mathcal{A} = \mathbf{R}\pi_{1*}\tau^*\pi_2^*\mathcal{A}$$
$$\xrightarrow{\sim} \mathbf{R}\pi_{1*}j_!j^*\pi_2^*\mathcal{A} \xrightarrow{\mathbb{1}_{\mathbf{R}\pi_{1*}}\varepsilon'\mathbb{1}_{\pi_2^*}} \mathbf{R}\pi_{1*}\pi_2^*\mathcal{A}.$$

On the stalk at the point 1, of the sheaf $\mathbf{R}^i\pi_{1*}\pi_2^*\mathcal{A}$, the map induced by γ is an isomorphism. On the stalk at the point 0, the map induced by γ is zero. An endomorphism of a constant sheaf over a connected base is constant. Thus, $\mathbf{R}^i\pi_{1*}\pi_2^*\mathcal{A}$ must be zero. Consequently, $\mathbf{R}\pi_{1*}\pi_2^*\mathcal{A} = 0$. □

1.3. The functor δ^* (resp. $\delta_!$) is left adjoint to δ_* (resp. $\delta^!$). In particular, there are canonical maps $\eta\colon \mathrm{id} \to \delta_*\delta^*$ and $\varepsilon\colon \delta_!\delta^! \to \mathrm{id}$. Applying $\mathbf{R}a_*$ (resp. $\mathbf{R}a_!$) we obtain maps $\mathbb{1}_{\mathbf{R}a_*}\eta\colon \mathbf{R}a_* \to \delta^*$ and $\mathbb{1}_{\mathbf{R}a_!}\varepsilon\colon \delta^! \to \mathbf{R}a_!$.

Proposition. *Let $\mathcal{A}$ be a $\mathbf{C}^*$-equivariant sheaf on V. Then $\mathbb{1}_{\mathbf{R}a_*}\eta\colon \mathbf{R}a_*\mathcal{A} \xrightarrow{\sim} \delta^*\mathcal{A}$ and $\mathbb{1}_{\mathbf{R}a_!}\varepsilon\colon \delta^! \xrightarrow{\sim} \mathbf{R}a_!$ are isomorphisms.*

Proof. Let $\mathcal{K}$ be the kernel of the morphism $\eta\colon \mathcal{A} \to \delta_*\delta^*\mathcal{A}$. Then $\mathcal{K}$ inherits a $\mathbf{C}^*$-equivariant structure from $\mathcal{A}$ and furthermore $\delta^*\mathcal{K} = 0$. As η is surjective, we have an exact sequence $0 \to \mathcal{K} \to \mathcal{A} \xrightarrow{\eta} \delta_*\delta^*\mathcal{A} \to 0$. This gives a distinguished triangle $\mathcal{K} \to \mathcal{A} \xrightarrow{\eta} \delta_*\delta^*\mathcal{A} \rightsquigarrow$. Applying $\mathbf{R}a_*$ and using the previous Lemma we get a distinguished triangle $0 \to \mathbf{R}a_*\mathcal{A} \xrightarrow{\mathbb{1}_{\mathbf{R}a_*}\eta} \delta^*\mathcal{A} \rightsquigarrow$. Thus, $\mathbb{1}_{\mathbf{R}a_*}\eta\colon \mathbf{R}a_*\mathcal{A} \xrightarrow{\sim} \delta^*\mathcal{A}$ is an isomorphism.

The second isomorphism is obtained from the first by Verdier duality. □

1.4. Let $\mathcal{T}$ be a triangulated category. An object $X \in \mathcal{T}$ is filtered by objects $Y_1, \dots, Y_n$ if there exists a sequence of objects $0 = X_0, X_1, \dots, X_n = X$ and distinguished triangles $X_{i-1} \to X_i \to Y_i \rightsquigarrow$.

Example. Let $\mathcal{A}$ be a bounded complex of sheaves. Then, in the derived category of sheaves, $\mathcal{A}$ is filtered by shifts of its cohomology sheaves.

1.5. We can now show:

Theorem ([Sp, §3])**.** *Let $\mathcal{A}^\cdot$ be a bounded complex of sheaves whose cohomology sheaves are $\mathbf{C}^*$-equivariant. Then $\mathbb{1}_{\mathbf{R}a_*}\eta\colon \mathbf{R}a_*\mathcal{A}^\cdot \xrightarrow{\sim} \delta^*\mathcal{A}^\cdot$ and $\mathbb{1}_{\mathbf{R}a_!}\varepsilon\colon \delta^!\mathcal{A}^\cdot \xrightarrow{\sim} \mathbf{R}a_!\mathcal{A}^\cdot$ are isomorphisms.*

Proof. By assumption, $\mathcal{A}^\cdot$ is filtered by objects $Y_1, \dots, Y_n$, where each object Y_i is the shift of a $\mathbf{C}^*$-equivariant sheaf. So, we have a sequence of objects $0 = X_0, X_1, \dots, X_n = \mathcal{A}^\cdot$ and distinguished triangles $X_{i-1} \to X_i \to Y_i \rightsquigarrow$. As $X_1 \simeq Y_1$, the previous Proposition gives that $\mathbb{1}_{\mathbf{R}a_*}\eta\colon \mathbf{R}a_*X_1 \xrightarrow{\sim} \delta^*X_1$. Proceeding by induction, we obtain that for all i, $\mathbb{1}_{\mathbf{R}a_*}\eta\colon \mathbf{R}a_*X_i \xrightarrow{\sim} \delta^*X_i$. In particular, $\mathbb{1}_{\mathbf{R}a_*}\eta\colon \mathbf{R}a_*\mathcal{A}^\cdot \xrightarrow{\sim} \delta^*\mathcal{A}^\cdot$.

The second isomorphism follows from the first by Verdier duality. □

References

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