The Orbit-Stabilizer Theorem

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An action of a group $G$ on a set $S$ is a map

$$G \times S \rightarrow S$$

notated by the juxtaposition $(g, s) \mapsto gs$, such that $1_Gs = s$ for all $s \in S$ and $(g_1g_2)s = g_1(g_2s)$ for all $g_1, g_2 \in G$ and $s \in S$. The orbit of a point $s \in S$ under the action is the set of points $O_s = \{gs \mid g \in G\}$. The stabilizer of $s$ is $\text{stab}(s) = \{g \in G \mid gs = s\}$.

**Theorem** (Orbit-Stabilizer). $|G| = |O_p||\text{stab}(p)|$

**Proof.** For every $x \in O_p$ define

$$H_x = \{g \mid gp = x, g \in G\}$$

Clearly for distinct $x, y H_x$ and $H_y$ are disjoint; as if $g \in H_x$ and $g \in H_y$ we have that $gp = x$ as well as $gp = y$ which gives us that $x = y$. Furthermore as sets

$$G = \bigcup_{x \in O_p} H_x$$

As clearly $\bigcup_{x \in O_p} H_x \subseteq G$ and if $g \in G$ we have that $gp = s$ for some $s \in S$ which gives us that $s \in O_p$ and hence $g \in H_s$, thus $G \subseteq \bigcup_{x \in O_p} H_x$.

Thus we have that

$$|G| = \sum_{x \in O_p} |H_x|$$

Note that $p \in O_p$ and $H_p = \text{stab}(p)$. We will show that $|H_x| = |H_p|$ for all $x \in O_p$ which will subsequently give us that

$$|G| = \sum_{x \in O_p} |H_x| = |O_p||\text{stab}(p)|$$

Pick some (fixed) $y \in H_x$ and define a map from $\text{stab}(p)$ to $H_x$ by

$$h \rightarrow yh$$

for $h \in \text{stab}(p)$. 
We need to show that this map is a bijection. Clearly the map is injective, to see surjectivity let $h \in H_x$ and consider the element $y^{-1}h$. Now $yp = x$ as $y \in H_x$, this implies that $y^{-1}x = p$ which gives us that
\[ y^{-1}hp = y^{-1}x = p \]
Thus, $y^{-1}h \in stab(p)$. Furthermore $y(y^{-1}h) = h$. So our map is surjective and we are done! \qed