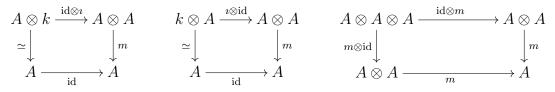
HOPF ALGEBRAS

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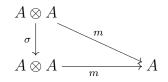
0.1. Notation. Let k be a field. For simplicity we will assume k to be algebraically closed. We will write \otimes for the usual tensor product over k. Further, for a k-vector space V, set $V^* = \operatorname{Hom}_k(V, k)$. All modules considered will be *left* modules unless otherwise stated.

0.2. Algebras and coalgebras. An *algebra* is a k-vector space with linear maps

called the multiplication and the unit respectively, such that the following diagrams commute



The commutativity of the third diagram is the usual associativity axiom. Define a linear map $\sigma: A \otimes A \to A \otimes A$ by $a \otimes b \mapsto b \otimes a$. Then A is commutative if the diagram



commutes.

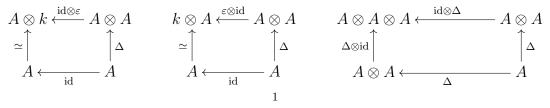
Suppose A and B are algebras with multiplication m_A and m_B respectively. Then $A \otimes B$ is an algebra with multiplication $m_{A \otimes B}$ defined to be the composite

That is $(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$.

A *coalgebra* is a k vector space with linear maps

$$\begin{array}{rcl} \Delta:A & \longrightarrow & A\otimes A, \\ \varepsilon:A & \longrightarrow & k, \end{array}$$

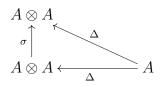
called the comultiplication and the counit respectively, such that the following diagrams commute



The commutativity of the third diagram is referred to as coassociativity. For $a \in A$ we will use Sweedler notation and write

$$\Delta(a) = \sum_{a} a_{(1)} \otimes a_{(2)}.$$

Let σ be as before, a coalgebra is said to be cocommutative if the diagram



commutes.

Let A be an algebra. Identify $(A \otimes A)^*$ with $A^* \otimes A^*$. This induces a coalgebra structure on A^* , namely

$$\Delta: A^* \to A^* \otimes A^*,$$
$$\varphi \mapsto (a \otimes a' \mapsto \varphi(aa')).$$

Dually, for a coalgebra C the coproduct induces an algebra structure on C^* . Namely, if $\varphi, \psi \in C^*$ then $(\varphi\psi)(x) = \sum_x \varphi(x_{(1)})\psi(x_{(2)})$, where $x \in C$.

Let C and D be coalgebras with coproduct Δ_C and Δ_D respectively. Then $C \otimes D$ is a coalgebra with comultiplication $\Delta_{C \otimes D}$ defined to be the composite

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\operatorname{id}_C \otimes \sigma \otimes \operatorname{id}_D} C \otimes D \otimes C \otimes D.$$

0.3. Convolution product. Suppose A is an algebra and C is a coalgebra. Identify $A \otimes C^*$ with $\operatorname{Hom}_k(C, A)$. This induces a multiplication *, called *convolution*, on $\operatorname{Hom}_k(C, A)$. Namely, for $\varphi, \psi \in \operatorname{Hom}_k(C, A)$

$$(\varphi * \psi)(x) = \sum_{x} \varphi(x_{(1)})\psi(x_{(2)}).$$

The identity of this ring is given by $\iota \circ \varepsilon$.

0.4. Hopf algebras. A Hopf algebra is a k-vector space A that is both an algebra and coalgebra such that

- (i) the comultiplication Δ and the counit ε are homomorphisms of algebras;
- (ii) the multiplication m and the unit i are homomorphisms of coalgebras;
- (iii) A is equipped with a bijective k-module map $S: A \to A$, called the antipode, such that the following diagrams commute

$$\begin{array}{cccc} A \otimes A \xrightarrow{S \otimes \mathrm{id}} A \otimes A & & A \otimes A \xrightarrow{\mathrm{id} \otimes S} A \otimes A \\ \Delta & & \downarrow^m & & \Delta \uparrow & & \downarrow^m \\ A \xrightarrow{\iota \circ \varepsilon} A & & A \xrightarrow{\iota \circ \varepsilon} A \end{array}$$

Proposition 0.1. The antipode is an algebra and coalgebra antiautomorphism, i.e. S(ab) = S(b)S(a) and $\sum_{a} S(a_{(2)}) \otimes S(a_{(1)}) = \sum_{S(a)} S(a)_{(1)} \otimes S(a)_{(2)}$.

Proof. Consider Hom $(A \otimes A, A)$ as an algebra under the convolution product. Let $M, S', S'' \in$ Hom $(A \otimes A, A)$ be the maps given by

$$M(a \otimes a') = aa',$$
 $S'(a \otimes a') = S(a')S(a)$ and $S''(a \otimes a') = S(aa').$

Then

$$M * S'(x \otimes y) = \sum_{x \otimes y} M((x \otimes y)_{(1)})S'((x \otimes y)_{(2)})$$

= $\sum_{x,y} M(x_{(1)} \otimes y_{(1)})S'(x_{(2)} \otimes y_{(2)})$
= $\sum_{x,y} x_{(1)}y_{(1)}S(y_{(1)})S(x_{(2)})$
= $\varepsilon(x)\varepsilon(y).$

Similarly

$$S'' * M(x \otimes y) = \sum_{x \otimes y} S''((x \otimes y)_{(1)}) M((x \otimes y)_{(2)})$$

= $\sum_{x,y} S''(x_{(1)} \otimes y_{(1)}) M(x_{(2)} \otimes y_{(2)})$
= $\sum_{x,y} S(x_{(1)}y_{(1)})(x_{(2)}y_{(2)})$
= $\sum_{xy} S((xy)_{(1)})(xy)_{(2)}$
= $\varepsilon(xy)$

Now consider $\operatorname{Hom}(A, A \otimes A)$ as an algebra under convolution. Let

$$\Delta * S' = \sum_{a} (a_{(1)} \otimes a_{(2)})(S(a_{(4)}) \otimes S(a_{(3)}))$$

$$= \sum_{a} a_{(1)}S(a_{(4)}) \otimes a_{(2)}S(a_{(3)})$$

$$= \sum_{a} a_{(1)}S(a_{(3)}) \otimes \varepsilon(a_{(2)})$$

$$= \sum_{a} a_{(1)}S(\varepsilon(a_{(2)})a_{(3)}) \otimes 1$$

$$= \sum_{a} a_{(1)}S(a_{(2)}) \otimes 1$$

$$= \varepsilon(a) \otimes 1.$$

Similarly

$$S'' * \Delta = \sum_{a} \Delta(S(a_{(1)}))\Delta(a_{(2)})$$

=
$$\sum_{a} \Delta(S(a_{(1)})a_{(2)})$$

=
$$\Delta(\varepsilon(a))$$

=
$$\varepsilon(a) \otimes 1$$

0.5. Some representation theory. Let A be a Hopf algebra throughout. We turn the field k into an A-module via

$$a \cdot 1 = \varepsilon(a), \qquad a \in A.$$

This is the *trivial representation* and by an abuse of notation is also denoted by k. The *adjoint representation* of A on itself is given by

ad :
$$A \otimes A \to A$$
,
 $a \otimes a' \mapsto \sum_{a} a_{(1)}a'S(a_{(2)}).$

The regular representation of A on itself is given by the multiplication of A. Let V be an A-module, then V^* is also an A-module via

$$a \cdot f(v) = f(S(a)v), \qquad a \in A, v \in V, f \in V^*.$$

Furthermore, given A-modules V and W, $V \otimes W$ is also an A-module via

$$a \cdot (v \otimes w) = \sum_{a} a_{(1)}v \otimes a_{(2)}w, \qquad a \in A, v \in V, w \in W.$$

We also give $\operatorname{Hom}_k(U, V)$ an A-module structure through the identification

$$V \otimes U^* \simeq \operatorname{Hom}_k(U, V)$$
 given by $v \otimes u^* \mapsto (f : u' \mapsto u^*(u')v).$

We note that it is implicit in this statement that for a module V, V^* will always be the restricted dual of V (i.e. functions with finite dimensional support).

Warning. In general $V \otimes U^*$ is not isomorphic to $U^* \otimes V$ as an A-module.

Let M be an A module. The *invariants* of M are elements of the submodule

$$M^{A} = \{ m \in M \mid am = \varepsilon(a)m \text{ for all } a \in A \}.$$

Remark 0.2. Taking invariants is a left exact functor on the category of A-modules. Further, this functor is representable, namely $M^A \simeq \operatorname{Hom}_A(k, M)$.

Lemma 0.3. Let M, N be A-modules. Then

$$\operatorname{Hom}_{A}(M, N) = \operatorname{Hom}_{k}(M, N)^{A}.$$

Proof. Suppose $\varphi \in \operatorname{Hom}_A(M, N)$, then

$$(a\varphi)(m) = \sum_{a} a_{(1)}\varphi(S(a_{(2)})m)$$
$$= \sum_{a} a_{(1)}S(a_{(2)})\varphi(m)$$
$$= \varepsilon(a)\varphi(m).$$

Conversely, suppose $\varphi \in \operatorname{Hom}_k(M, N)^A$ and $a \in A$ then

$$\begin{aligned} a\varphi(m) &= \sum_{a} a_{(1)} \varepsilon(a_{(2)})\varphi(m) \\ &= \sum_{a} a_{(1)} \varphi(\varepsilon(a_{(2)})m) \\ &= \sum_{a} a_{(1)} \varphi(S(a_{(2)})a_{(3)}m) \\ &= \sum_{a} (a_{(1)}\varphi)(a_{(2)}m) \\ &= \sum_{a} \varepsilon(a_{(1)})\varphi(a_{(2)}m) \\ &= \sum_{a} \varphi(\varepsilon(a_{(1)})a_{(2)}m) \\ &= \varphi(am). \end{aligned}$$

Write $\operatorname{Ext}_A(M, -)$ for the right derived functors of $\operatorname{Hom}_A(M, -)$.

Corollary 0.4. Let M, N be A-modules, then there is a natural isomorphism

$$\operatorname{Ext}_{A}^{i}(M, N) \simeq \operatorname{Ext}_{A}^{i}(k, \operatorname{Hom}_{k}(M, N))$$

Proof. We have the following isomorphism of functors

$$\operatorname{Hom}_A(M, -) \simeq \operatorname{Hom}_k(M, -)^A \simeq \operatorname{Hom}_A(k, \operatorname{Hom}_k(M, -)).$$

Thus, the corresponding derived functors are isomorphic.

0.6. An example: Group algebras. Let G be a finite group and let $\mathbb{C}[G]$ be its group algebra over \mathbb{C} . Then $A = \mathbb{C}[G]$ is a Hopf algebra with

$$\Delta(g) = g \otimes g,$$
 $S(g) = g^{-1}$ and $\varepsilon(g) = 1,$ $g \in G.$

Set

$$\Omega = \frac{1}{|G|} \sum_{g \in G} g.$$

The element Ω is a central in A. Furthermore, $g\Omega = \Omega g = \Omega$ for all $g \in G$. Thus, letting k denote the trivial module, we have that $\Omega A \simeq k^{\oplus |G|}$. Furthermore, as Ω is a central idempotent, $A \simeq \Omega A \oplus (1 - \Omega)A$. Hence, k is a projective module. Thus, if M is a finite dimensional A-module, then $\operatorname{Ext}^{i}(k, M) = 0$ for i > 0. Consequently, if M and N are finite

dimensional A-modules then by Corollary 0.4 we have that $\operatorname{Ext}^{i}(M, N) = 0, i > 0$. In particular all finite dimensional A-modules are completely reducible.

Remark 0.5. The above arguments generalize to $\mathbb{F}[G]$, where |G| does not divide char \mathbb{F} .

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