THE GROTHENDIECK GROUP

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If C is an abelian category, we write K(C) for the *Grothendieck group* of C. This is the quotient of the free abelian group with generators the objects $M \in C$, by the subgroup generated by elements $M_1 - M_2 + M_3$ for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

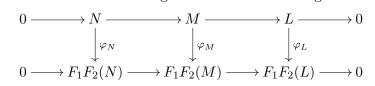
in \mathcal{C} . If objects in \mathcal{C} have finite length and unique composition factors we write $K(\mathcal{C})^*$ for the topological dual of $K(\mathcal{C})$; i.e. for the linear functions $f: K(\mathcal{C}) \to \mathbb{Z}$ such that f(M) = 0 for all but finitely many isomorphism classes of irreducible objects $M \in \mathcal{C}$. If $L \in \mathcal{C}$, let's write [L] for its image in $K(\mathcal{C})$. Then as L runs through the irreducible objects in \mathcal{C} , the elements [L] form a basis of $K(\mathcal{C})$, and the functions

$$\delta_L : K(\mathcal{C}) \to \mathbb{Z} \qquad \delta_L(L') = \begin{cases} 0 & \text{if } L \not\cong L', \ L' \text{ irreducible} \\ 1 & \text{if } L \cong L' \end{cases}$$

form a basis of $K(\mathcal{C})^*$. More generally, if $M \in \mathcal{C}$ and L is an irreducible object in \mathcal{C} write [M : L] for the multiplicity of L in a Jordan-Holder series of M, and extend this bilinearly to $[:]: K(\mathcal{C}) \times K(\mathcal{C}) \to \mathbb{Z}$. Then for any $M \in K(\mathcal{C})$, write $\delta_M : K(\mathcal{C}) \to \mathbb{Z}$ for the function $N \mapsto [N : M]$. Now, if $F : \mathcal{C} \to \mathcal{C}'$ is an exact functor of abelian categories, we get an induced \mathbb{Z} -linear map $F : K(\mathcal{C}) \to K(\mathcal{C}')$, and we can define its transpose $F^* : K(\mathcal{C}')^* \to K(\mathcal{C})^*$ by $F^*f = fF$. Write $K(\mathcal{C})_{\mathbb{Q}} = K(\mathcal{C}) \otimes \mathbb{Q}$. As $K(\mathcal{C})$ is a torsion free \mathbb{Z} -module, $K(\mathcal{C})_{\mathbb{Q}}$ is a \mathbb{Q} -vector space with distinguished sublattice $K(\mathcal{C}) \subset K(\mathcal{C})_{\mathbb{Q}}$.

Proposition 0.0.1. Let $F : C_1 \to C_2$ and $G : C_2 \to C_1$ be mutually adjoint exact functors between two Artinian abelian categories C_1 and C_2 . Then F and G are mutual equivalences of categories if and only if they define mutually inverse isomorphisms on the level of Grothendieck groups.

Proof. For $M \in C_2$, let φ_M denote the adjunction map $M \to F_1F_2(M)$. Suppose $L \in C_2$ is irreducible. Since $[L] = [F_1F_2(L)]$, we have that φ_L is an isomorphism. Now proceed by induction on length. For arbitrary $M \in C_2$ we have an exact sequence $0 \to N \to M \to L \to 0$, with N of lower length than ${\cal M}$ and ${\cal L}$ irreducible. This gives a commutative diagram



The map φ_N is an isomorphism by induction; φ_L is an isomorphism by the previous argument. This forces φ_M to be an isomorphism.

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