Cesàro Summability of Fourier Series

February 25, 2005

1 Preliminaries

Throughout this abstract we will use the following convention:

\[ D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt) = \begin{cases} \frac{\sin((n+\frac{1}{2})t)}{2\sin(t)} & \text{for } t \neq 2m\pi \text{ rational} \\ n + 1/2 & \text{for } t = 2m\pi \end{cases} \]

i.e. \( D_n(t) \) is Dirichlet’s kernel.

Furthermore we will let \( L(I) \) denote the set of Lebesgue-integrable functions on an interval \( I \).

**Theorem 1.1.** For every real \( x \neq 2m\pi \) (\( m \) is an integer), we have

\[ \sum_{k=1}^{n} e^{ikx} = e^{ix} \frac{1 - e^{inx}}{1 - e^{ix}} \]

and that

\[ \sum_{k=0}^{n} \sin((2k+1)x) = \frac{\sin^2(nx)}{\sin(x)} \]

**Proof.** \( (1 - e^{ix}) \sum_{k=1}^{n} e^{ikx} = \sum_{k=1}^{n} (e^{ikx} - e^{i(k+1)x}) = e^{ix} - e^{i(n+1)x} \). This gives us the first identity, the second one is obtained by comparing the real and imaginary parts of the first one. \( \square \)

**Theorem 1.2.** Assume that \( f \in L([0, 2\pi]) \) and suppose that \( f \) is periodic with period \( 2\pi \). Let \( \{s_n\} \) denote the sequence of partial sums of the Fourier series generated by \( f \) say

\[ s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \]
Then we have the integral representation

\[ s_n(x) = \frac{2}{\pi} \int_{0}^{\pi} f(x+t) + f(x-t) D_n(t) \, dt \]

Proof. Substituting the integral representation of the Fourier coefficients in the formula for \( s_n(x) \) we have that

\[ s_n(x) = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} \left( \cos(kt) \cos(kx) + \sin(kt)(kx) \right) \right\} \, dt \]

\[ = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} \cos(k(t-x)) \right\} \, dt = \frac{1}{\pi} \int_{0}^{2\pi} f(t) D_n(t-x) \, dt \]

Since both \( f \) and \( D_n \) are periodic with period \( 2\pi \), we can replace the interval of integration by \([x - \pi, x + \pi]\) and then make a translation by \( u = t - x \) to get

\[ s_n(x) = \frac{1}{\pi} \int_{0}^{2\pi} f(t) D_n(t-x) \, dt \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) D_n(u) \, du \]

Using the fact that \( D_n(-u) = D_n(u) \) we get the required formula.

\[ \square \]

2 Cesàro Summability of Fourier Series

Continuity of a function is usually not strong enough to say anything conclusively about the convergence of its Fourier series. In 1873, Du Bois Reymond gave an example of a function, continuous throughout the interval \([0, 2\pi]\), whose Fourier series fails to converge on an uncountable subset of the same interval. On the other hand, continuity does suffice to establish what is called Cesàro summability of the series. The main theorem here is due to Fejér.

**Theorem 2.1.** Assume that \( f \in L([0, 2\pi]) \) and suppose that \( f \) is periodic with period \( 2\pi \). Let \( s_n \) denote the nth partial sum of the Fourier series generated by \( f \) and let

\[ \sigma_n(x) = \frac{s_0(x) + s_1(x) + \ldots + s_{n-1}(x)}{n} \]

Then we have the integral representation

\[ \sigma_n(x) = \frac{1}{n\pi} \int_{0}^{\pi} f(x+t) + f(x-t) \frac{\sin^2 \left( \frac{nt}{2} \right)}{\sin^2 \left( \frac{x}{2} \right)} \, dt \]
Proof. The result is obtained immediately by using the integral representation for \( s_n(x) \) obtained in Theorem 1.2 and using the second identity from Theorem 1.1.

NOTE. If we apply the above theorem to the constant function whose value is 1 at each point we find \( \sigma_n(x) = s_n(x) = 1 \) for each \( n \) and hence we get that

\[
\sigma_n(x) - s = \frac{1}{n\pi} \int_0^\pi \left\{ \frac{f(x+t) + f(x-t)}{2} - s \right\} \frac{\sin^2 \left( \frac{nt}{2} \right)}{\sin^2 \left( \frac{t}{2} \right)} \, dt
\]

If we could choose the value of \( s \) such that the integral on the right tends to 0 as \( n \to \infty \), it will follow that \( \sigma_n(x) \to s \) as \( n \to \infty \). The next result shows us that it suffices to take \( s = \frac{[f(x+) + f(x-)]}{2} \).

Theorem 2.2 (Fejér). Assume that \( f \in L([0, 2\pi]) \) and suppose that \( f \) is periodic with period \( 2\pi \). Define a function \( s \) by

\[
s(x) = \lim_{t \to 0^+} \frac{f(x+t) + f(x-t)}{2} - s(x)
\]

whenever the limit exists. Then for each \( x \) for which \( s(x) \) is defined, the Fourier series generated by \( f \) is Cesàro summable. That is we have

\[
\lim_{n \to \infty} \sigma_n(x) = s(x)
\]

where \( \{\sigma_n\} \) is the sequence of arithmetic means defined earlier. If in addition, \( f \) is continuous on \([0, 2\pi]\), then the sequence \( \{\sigma_n\} \) converges uniformly to \( f \) on \([0, 2\pi]\).

Proof. Let \( g_x(t) = \frac{[f(x+t) + f(x-t)]}{2} - s(x) \), whenever \( s(x) \) is defined. Then \( g_x(t) \to 0 \) as \( t \to 0^+ \). Therefore given any \( \epsilon > 0 \) there is a \( 0 < \delta < \pi \) such that \( |g_x(t)| < \epsilon/2 \) whenever \( 0 < t < \delta \). Note that \( \delta \) depends on \( x \) as well as on \( \epsilon \). However, if \( f \) is continuous on \([0, 2\pi]\), then \( f \) is uniformly continuous on the same and there exists a \( \delta \) which serves equally well for all \( x \) on the same interval. Now we use the integral representation obtained in the previous theorem and divide the interval of integration into two subintervals \([0, \delta]\) and \([\delta, \pi]\). On \([0, \delta]\) we have

\[
\left| \frac{1}{n\pi} \int_0^\delta g_x(t) \frac{\sin^2 \left( \frac{nt}{2} \right)}{\sin^2 \left( \frac{t}{2} \right)} \, dt \right| \leq \frac{\epsilon}{2n\pi} \int_0^\pi \frac{\sin^2 \left( \frac{nt}{2} \right)}{\sin^2 \left( \frac{t}{2} \right)} \, dt = \frac{\epsilon}{2}
\]

On \([\delta, \pi]\) we have that

\[
\left| \frac{1}{n\pi} \int_\delta^\pi g_x(t) \frac{\sin^2 \left( \frac{nt}{2} \right)}{\sin^2 \left( \frac{t}{2} \right)} \, dt \right| \leq \frac{1}{n\pi \sin^2(\delta/2)} \int_\delta^\pi |g_x(t)| \, dt \leq \frac{I(x)}{n\pi \sin^2(\delta/2)}
\]
where \( I(x) = \int_0^\pi |g_x(t)| dt \). Now we can choose \( n \) large enough so that the expression on the right is smaller than \( \epsilon/2 \). Thus there exists a \( N \) such that for \( n > N \)

\[
|\sigma_n(x) - s(x)| < \epsilon
\]

In other words, \( \sigma_n(x) \to s(x) \) as \( n \to \infty \).

Now if \( f \) is continuous then it is bounded on \([0, 2\pi]\) and we can replace \( I(x) \) by a constant. The resulting \( N \) is then independent of \( x \) giving us uniform convergence.

\[ \square \]

### 3 Consequences

**Theorem 3.1.** Let \( f \) be continuous on \([0, 2\pi]\) and periodic with period \( 2\pi \). Let \( \{s_n\} \) be as before, and let \( a_n, b_n \) be the Fourier coefficients of \( f \). Then we have:

a) l.i.m. \( n \to \infty \) = \( f \) on \([0, 2\pi]\).

b) \( \frac{1}{\pi} \int_0^{2\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \) (Parseval’s formula)

c) The Fourier series for \( f \) can be integrated term by term, the integrated series being uniformly convergent, even if the original Fourier series diverges.

d) If the Fourier series of \( f \) converges for some \( x \), then it converges to \( f(x) \).

**Proof.** Part (a) follows by the theorem on best approximation in the mean. (b) follows from (a) and Parseval’s inequality. (c) follows from (a) using the Cauchy-Schwarz inequality for integrals. (d) is trivial. \( \square \)

**Theorem 3.2.** Let \( f \) be a real valued and continuous on \([a, b]\). Then for every \( \epsilon > 0 \) there is a polynomial \( p \) (which may depend on \( \epsilon \)) such that

\[ |f(x) - p(x)| < \epsilon \quad \text{for every } x \in [a, b] \]

**Proof.** First define a new function \( g \) that is a parametrization of \( f \) that changes the domain to \([0, 2\pi]\) and has period \( 2\pi \). Then use Fejér’s theorem to approximate the function using finite Fourier sums. Each term of this finite sum is a trigonometric function which is analytic and can thus be approximated by polynomials, thus we obtain a polynomial \( p \) that suitably approximates \( g \), now parametrise \( p \) to obtain the appropriate approximation for \( f \). \( \square \)