Cesàro Summability of Fourier Series

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1 Preliminaries

Throughout this abstract we will use the following convention:

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt) = \begin{cases} \frac{\sin((n+\frac{1}{2})t)}{2\sin(\frac{t}{2})} & \text{for } t \neq 2m\pi \text{ rational} \\ n + 1/2 & \text{for } t = 2m\pi \end{cases}$$

i.e. $D_n(t)$ *is Dirichlet's kernel.*

Furthermore we will let $L(I)$ *denote the set of Lebesgue-integrable functions on an interval* I*.*

Theorem 1.1. *For every real* $x \neq 2m\pi$ *(*m *is an integer), we have*

$$\sum_{k=1}^{n} e^{ikx} = e^{ix} \frac{1 - e^{inx}}{1 - e^{ix}}$$

and that

$$\sum_{k=0}^{n} \sin((2k+1)x) = \frac{\sin^2(nx)}{\sin(x)}$$

Proof. $(1 - e^{ix}) \sum_{k=1}^{n} e^{ikx} = \sum_{k=1}^{n} (e^{ikx} - e^{i(k+1)x}) = e^{ix} - e^{i(n+1)x}$. This gives us the first identity, the second one is obtained by comparing the real and imaginary parts of the first on. □

Theorem 1.2. *Assume that* $f \in L([0, 2\pi])$ *and suppose that* f *is periodic with period* 2π*. Let* $\{s_n\}$ *denote the sequence of partial sums of the Fourier series generated by* f *say*

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$$

Then we have the integral representation

$$s_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} D_n(t)\, dt$$

Proof. Substituting the integral representation of the Fourier coefficients in the formula for $s_n(x)$ we have that

$$\begin{aligned} s_n(x) &= \frac{1}{\pi} \int_0^{2\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} (\cos(kt)\cos(kx) + \sin(kt)(kx) \right\} dt \\ &= \frac{1}{\pi} \int_0^{2\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} \cos(k(t-x)) \right\} dt = \frac{1}{\pi} \int_0^{2\pi} f(t) D_n(t-x)\, dt \end{aligned}$$

Since both f and D_n are periodic with period 2π, we can replace the interval of integration by $[x-\pi, x+\pi]$ and then make a translation by $u = t - x$ to get

$$\begin{aligned} s_n(x) &= \frac{1}{\pi} \int_0^{2\pi} f(t) D_n(t-x)\, dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) D_n(u)\, du \end{aligned}$$

Using the fact that $D_n(-u) = D_n(u)$ we get the required formula. □

2 Cesàro Summability of Fourier Series

Continuity of a function is usually not strong enough to say anything conclusively about the convergence of its Fourier series. In 1873, Du Bois Reymond gave an example of a function, continuous throughout the interval $[0, 2\pi]$, whose Fourier series fails to converge on an uncountable subset of the same interval. On the other hand , continuity does suffice to establish what is called Cesàro summability of the series. The main theorem here is due to Fejér.

Theorem 2.1. *Assume that $f \in L([0, 2\pi])$ and suppose that f is periodic with period 2π. Let s_n denote the nth partial sum of the Fourier series generated by f and let*

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + \ldots + s_{n-1}(x)}{n}$$

Then we have the integral representation

$$\sigma_n(x) = \frac{1}{n\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} \frac{\sin^2\left(\frac{nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}\, dt$$

Proof. The result is obtained immediately by using the integral representation for $s_n(x)$ obtained in Theorem 1.2 and using the second identity from Theorem 1.1 $\square$

NOTE. If we apply the above theorem to the constant function whose value is 1 at each point we find $\sigma_n(x) = s_n(x) = 1$ for each n and hence we get that

$$\sigma_n(x) - s = \frac{1}{n\pi}\int_0^\pi \left\{\frac{f(x+t)+f(x-t)}{2} - s\right\}\frac{\sin^2\left(\frac{nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}\,dt$$

If we could choose the value of s such that the integral on the right tends to 0 as $n \to \infty$, it will follow that $\sigma_n(x) \to s$ as $n \to \infty$. The next result shows us that it suffices to take $s = [f(x+) + f(x-)]/2$.

Theorem 2.2 (Fejér). *Assume that $f \in L([0, 2\pi)]$ and suppose that f is periodic with period 2π. Define a function s by*

$$s(x) = \lim_{t\to 0^+}\frac{f(x+t)+f(x-t)}{2}$$

whenever the limit exists. Then for each x for which $s(x)$ is defined, the Fourier series generated by f is Cesàro summable. That is we have

$$\lim_{n\to\infty}\sigma_n(x) = s(x)$$

where $\{\sigma_n\}$ is the sequence of arithmetic means defined earlier. If in addition, f is continuous on $[0, 2\pi]$, then the sequence $\{\sigma_n\}$ converges uniformly to f on $[0, 2\pi]$.

Proof. Let $g_x(t) = [f(x+t) + f(x-t)]/2 - s(x)$, whenever $s(x)$ is defined. Then $g_x(t) \to 0$ as $t \to 0^+$. Therefore given any $\epsilon > 0$ there is a $0 < \delta < \pi$ such that $|g_x(t)| < \epsilon/2$ whenever $0 < t < \delta$. Note that δ depends on x as well as on ϵ. However, if f is continuous on $[0, 2\pi]$, then f is uniformly continuous on the same and there exists a δ which serves equally well for all x on the same interval. Now we use the integral representation obtained in the previous theorem and divide the interval of integration into two subintervals $[0, \delta]$ abd $[\delta, \pi$. On $[0, \delta]$ we have

$$\left|\frac{1}{n\pi}\int_0^\delta g_x(t)\frac{\sin^2\left(\frac{nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}\,dt\right| \le \frac{\epsilon}{2n\pi}\int_0^\pi \frac{\sin^2\left(\frac{nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}\,dt = \frac{\epsilon}{2}$$

On $[\delta, \pi]$ we have that

$$\left|\frac{1}{n\pi}\int_\delta^\pi g_x(t)\frac{\sin^2\left(\frac{nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}\,dt\right| \le \frac{1}{n\pi\sin^2(\delta/2)}\int_\delta^\pi |g_x(t)|\,dt \le \frac{I(x)}{n\pi\sin^2(\delta/2)}$$

where $I(x) = \int_0^{\pi} |g_x(t)|\, dt$. Now we can choose n large enough so that the expression on the right is smaller than $\epsilon/2$. Thus there exists a N such that for $n > N$

$$|\sigma_n(x) - s(x)| < \epsilon|$$

In other words, $\sigma_n(x) \to s(x)$ as $n \to \infty$.
Now if f is continuous then it is bounded on $[0, 2\pi]$ and we can replace $I(x)$ by a constant. The resulting N is then independent of $x$giving us uniform convergence. □

3 Consequences

Theorem 3.1. *Let f be continuous on $[0, 2\pi]$ and periodic with period 2π. Let $\{s_n\}$ be as before, and let a_n, b_n be the Fourier coefficients of f. Then we have:*

a) $l.i.m._{n\to\infty} = f$ on $[0, 2\pi]$.

b) $\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ (Parseval's formula)

c) The Fourier series for f can be integrated term by term, the integrated series being uniformly convergent, even if the original Fourier series diverges.

d) If the Fourier series of f converges for some x, then it converges to $f(x)$.

Proof. Part (a) follows by the theorem on best approximation in the mean. (b) follows from (a) and Parseval's inequality. (c) follows from (a) using the Cauchy-Schwarz inequality for integrals. (d) is trivial. □

Theorem 3.2. *Let f be a real valued and continuous on $[a, b]$. Then for every $\epsilon > 0$ there is a polynomial p (which may depend on ϵ) such that*

$$|f(x) - p(x| < \epsilon \quad \textit{for every } x \in [a, b]$$

Proof. First define a new function g that is a parametrization of f that changes the domain to $[0, 2\pi]$ and has period 2π. Then use Fejér's theorem to approximate the function using finite Fourier sums. Each term of this finite sum is a trigonometric function which is analytic and can thus be approximated by polynomials, thus we obtain a polynomial p that suitably approximates g, now parametrise p to obtain the appropriate approximation for f. □