Categories and Functors

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1 Categories

A *category* $\mathcal{C}$ consists of a collection of *objects* $\mathrm{Ob}(\mathcal{C})$; and for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ a set $\mathrm{Hom}(A, B)$ called the set of *morphisms* of A into B; and for any three objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a law of composition (i.e. a map)

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C)$$

satisfying the following axioms

1. Two set $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A', B')$ are disjoint unles $A = A'$ and $B = B'$ in which case they are equal.

2. For each object A of $\mathcal{C}$ there is a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ which acts as left and right identity for the elements of $\mathrm{Hom}(B, A)$ and $\mathrm{Hom}(A, B)$ respectively, for all objects B in $\mathcal{C}$.

3. The law of composition is associative (when defined), i.e. given $f \in \mathrm{Hom}(A, B), g \in \mathrm{Hom}(B, C), h \in \mathrm{Hom}(C, D)$ then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

   for all objects A, B, C, D of $\mathcal{C}$.

A morphism $f : A \longrightarrow B$ is called an *isomorphism* if there exists a morphism $g : B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. If $A = B$ we also say that the isomorphism is an *automorphism*. A morphism of an object A into itself is called an *endomorphism*. For an object A in $\mathcal{C}$ we denote by $\mathrm{Aut}(A)$ the set of automorphisms of A, and by $\mathrm{End}(A)$ the set of endomorphisms of A.

Let G be a group and let $\mathcal{C}$ be a category and $A \in \mathrm{Ob}(\mathcal{C})$. By an *operation* of G on A we shall mean a homomorphism

$$\varphi : G \longrightarrow \mathrm{Aut}(A).$$

An operation of a group G on an object A is also called a *representation* of G on A.

2 Universal Objects

Let $\mathcal{C}$ be a category. An object U of $\mathcal{C}$ is called *universally attracting* if there exists a unique morphism of each object of $\mathcal{C}$ into U, and is called *universally repelling* if for every object of $\mathcal{C}$ there exists a unique morphism of U into that object. Since a universal object admits the identity morphism into itself, it is clear that if U and U' are two universal objects in $\mathcal{C}$, then there exists a unique isomorphism between them.

3 Products and Coproducts

Let $\mathcal{C}$ be a category and let A, B be objects of $\mathcal{C}$. By a *product* of A, B in $\mathcal{C}$ one means a triple (P, f, g) consisting of an object P in $\mathcal{C}$ and two morphisms f, g

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

satisfying the following condition. Given two morphisms

$$\varphi : C \longrightarrow A \qquad \text{and} \qquad \psi : C \longrightarrow B$$

in $\mathcal{C}$, there exists a unique morphism $h : C \longrightarrow P$ which makes the following diagram commutative

$$\begin{tikzcd} & C \arrow[d, "h"] \arrow[ddl, "\varphi"'] \arrow[ddr, "\psi"] & \\ & P \arrow[dl, "f"] \arrow[dr, "g"'] & \\ A & & B \end{tikzcd}$$

More generally, given a family of objects $\{A_i\}_{i \in I}$ in $\mathcal{C}$, a product for this family consists of $\{P, \{f_i\}_{i \in I}\}$ where P is an object in $\mathcal{C}$ and $\{f_i\}_{i \in I}$ is a family of morphisms

$$f_i : P \longrightarrow A_i,$$

satisfying the following condition; given a family of morphisms

$$g_i : C \longrightarrow A_i,$$

there exists a unique morphism $h : C \longrightarrow P$ such that $f_i \circ h = g_i$ for all i.

Let A, B be objects of a category $\mathcal{C}$. We note that the product of A, B is universal in the category whose objects consist of pairs of morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ in $\mathcal{C}$, and whose morphisms are described as follows. Let $f' : C' \longrightarrow A$ and $g' : C' \longrightarrow B$ be another pair, then a morphism from the first pair to the second is a morphism $h : C \longrightarrow C'$ in $\mathcal{C}$, making the following diagram commutative

$$\begin{tikzcd}
& C \arrow[d, "h"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& C' \arrow[dl, "f'"] \arrow[dr, "g'"'] & \\
A & & B
\end{tikzcd}$$

The situation is similar for the product of a family $\{A_i\}_{i\in I}$.

Example. Let $\mathcal{C}$ be the category of sets and let $\{A_i\}_{i\in I}$ be a family of sets. Let $A = \prod_{i\in I} A_i$ be their Cartesian product and let $p_i : A \longrightarrow A_i$ be the projection on the ith factor. Then $(A, \{P_i\}_{i\in I})$ satisfy the requirements of a product in the category of sets.

Example. Let $\{G_i\}_{i\in I}$ be a family of groups, and let $G = \prod G_i$ be their direct product. Let $p_i : G \longrightarrow G_i$ be the projection homomorphism. Then these constitute a product of the family in the category of groups.

Let $\{A_i\}_{i\in I}$ be a family of objects in a category $\mathcal{C}$. By their *coproduct* one means a pair $\{S, \{f_i\}_{i\in I}\}$ consisting of an object S and a family of morphisms

$$f_i : A_i \longrightarrow S,$$

satisfying the following property. Given a family of morphisms $\{g_i : A_i \longrightarrow C\}$ there exists a unique morphism $h : S \longrightarrow C$ such that $h \circ f_i = g_i$ for all i. The coproduct of a family $\{A_i\}$ will also be denoted by $\coprod A_i$ and similarly the coproduct of two objects A, B will also be denoted by $A \coprod B$. The coproduct of A and B is universal in the category of families maps from A and B into a single object.

Example. Let $\mathcal{S}$ be the category of sets. Then coproducts exist in this category. Let S and S' be sets. Let T be a set having the same cardinality as S' and disjoint from S. Let $f_1 : S \longrightarrow S$ be the identity, and $f_2 : S' \longrightarrow T$ be a bijection. Let U be the union of S and T. Then (U, f_1, f_2) is a coproduct for S and S'.

Example. Let cS_0 be the category of pointed sets. Its objects consist of pairs (S, x) where S is a set and x an element of S. A morphism of (S, x) into (S', x') in this category is a map $g : S \longrightarrow S'$ such that $g(x) = x'$. The coproduct in this category can be constructed as follows. Let T be a set with the same cardinality as S' and such that $T \cap S = \{x\}$. Let $U = S \cup T$, and let

$$f_1 : (S, x) \longrightarrow (U, x)$$

be the map which induces the identity on S. Let

$$f_2 : (S', x') \longrightarrow (U, x)$$

be the map sending x' to x and inducing a bijection of $S' - \{x'\}$ on $T - \{x\}$. Then the triple (U, f_1, f_2) is a coproduct for (S, x) and (S', x') in $\mathcal{S}_0$.

4 Fiber products, coproducts, pull-backs and push-outs

Let $\mathcal{C}$ be a category. Let Z be an object of $\mathcal{C}$. Then we have a new category of objects over Z, denoted by $\mathcal{C}_Z$. The objects of $\mathcal{C}_Z$ are morphisms:

$$f : X \longrightarrow Z \qquad \text{in } \mathcal{C}.$$

A morphism from f to $g : Y \longrightarrow Z$ in $\mathcal{C}_Z$ is merely a morphism $h : X \longrightarrow Y$ in $\mathcal{C}$ which makes the following diagram commute

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

A product in $\mathcal{C}_Z$ is called the *fiber product* of f and g in $\mathcal{C}$ and is denoted by $f \times_Z g$, together with its natural morphisms of X, Y over Z, which are sometimes not denoted by anything , but which we denote by p_1 and p_2.

$$\begin{tikzcd} & X \times_Z Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Example. Fiber products exist in the category of abelian groups. The fibered product of two homomorphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ is the subgroup of $X \times Y$ consisting of all pairs (x, y) such that $f(x) = g(y)$.

In the fiber product diagram, one calls p_1, the *pull-back* of g by f, and p_2 the pull-back of f by g. The fiber product satisfies the following universal property: Given any object T in $\mathcal{C}$ and morphisms making the following diagram commutative

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

there exists a unique morphism $T \longrightarrow X \times_Z Y$ making the following diagram commutative

$$\begin{tikzcd}
& T \arrow[dl] \arrow[d] \arrow[dr] & \\
& X \times_Z Y \arrow[l] \arrow[r] & \\
X & & Y
\end{tikzcd}$$

Dually we have the notion of *coproduct* in the category of morphisms $f : Z \longrightarrow X$ with a fixed object Z as the object of departure of the morphisms. This category is denoted by $\mathcal{C}^Z$. We reverse the arrows in the preceding discussion. Given two objects f and $g : Z \longrightarrow Y$ in $\mathcal{C}^Z$ we have their coproduct $X \coprod_Z Y$ with morphisms q_1 and q_2 making the following diagram commutative

$$\begin{tikzcd}
& X \coprod\limits_Z Y \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

satisfying the dual universal property of the fiber product. We call it the *fibered coproduct.* We call q_1 the *push-out* of f by f, and q_2 the push-out of f by g.

Example. Fibered coproducts exist in the category of abelian groups. The coproduct of two homomorphisms $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ is the factor group $X \oplus Y/W$ where W is the subgroup of $X \oplus Y$ consisting of all elements $(f(x), -g(z))$ with $z \in Z$.

5 Functors

Let $\mathcal{C}, \mathcal{D}$ be categories. A *covariant functor* F of $\mathcal{C}$ into $\mathcal{D}$ is a rule which to each object X in $\mathcal{C}$ asssociates an object $F(X)$ in $\mathcal{D}$, and to each morphism $f : X \longrightarrow Y$ associates a morphism $F(f) : F(X) \longrightarrow F(Y)$ such that:

1. For all X in $\mathcal{C}$ we have $F(\mathrm{id}_X) = id_{F(X)}$.
2. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms of $\mathcal{C}$ then

$$F(g \circ f) = F(g) \circ F(f).$$

We also have the notion of a *contravariant functor* that has essentially the same definition but 'reverses all arrows', i.e. to each morphism $f : X \longrightarrow Y$ the contravariant functor associates a morphism

$$F(f) : F(B) \longrightarrow F(A)$$

going in the opposite direction, such that if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are morphisms in $\mathcal{C}$, then

$$F(g \circ f) = F(f) \circ F(g).$$

Example. To each group G associate its set (stripped of the group structure) to obtain a covariant functor from the category of groups into the category of sets, provided we associate with each group homomorphism itself, viewed only as a set theoretic map. Such a functor is called a *stripping* functor or *forgetful* functor.

Example. Consider the category of abelian groups. Fix an abelian group G. The association $X \longmapsto \mathrm{Hom}(X, G)$ is a contravariant functor from this category to itself. The association $X \longmapsto \mathrm{Hom}(G, X)$ is a covariant functor from the category to itself.

Example. Let $\mathcal{C}$ be a category and G a fixed object in $\mathcal{C}$. Let $M_G(X) = \mathrm{Hom}(G, X)$ for any object X of $\mathcal{C}$. If $\varphi : X \longrightarrow X'$ is a morphism, let

$$\begin{array}{rccc} M_G(\varphi): & \mathrm{Hom}(G, X) & \longrightarrow & \mathrm{Hom}(G, X'), \\ & g & \longmapsto & \varphi \circ g \end{array}$$

for any $g \in \mathrm{Hom}(G, X)$. So M_G is a covariant functor from $\mathcal{C}$ to the category of sets.

Similarly, for each object Y of $\mathcal{C}$ we have a contravariant functor M^G from $\mathcal{C}$ to the category of sets given by $M^G(Y) = \mathrm{Hom}(Y, G)$. M_G and M^G are called *representation functors.*

Let $\mathcal{C}, \mathcal{D}$ be two categories. The functors of $\mathcal{C}$ into $\mathcal{D}$ (say covariant, and in one variable) may be viewed as objects of a category, whose morphisms (called natural transformations) are given as follows. Let F, G be two such functors. A *natural transformation* $H : F \longrightarrow G$ is a rule which to each object X of $\mathcal{C}$ associates a morphism

$$H_X : F(X) \longrightarrow G(X)$$

such that for any morphism $f : X \longrightarrow Y$ the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "H_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "H_Y"'] & G(Y) \end{tikzcd}$$

We therefore have the notion of isomorphisms of functors. A functor is *representable* if it is isomorphic to a representation functor.

6 Adjoint functors

If $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ are functors, then we say that F is a *left adjoint* for G (equivalently G is a *right adjoint* for F) if there is a natural isomorphism $\alpha : \mathrm{Hom}_{\mathcal{C}}(-, G(-)) \cong$

$\mathrm{Hom}_{\mathcal{D}}(F(-),-)$. That is, for every pair of objects X of C and Y of D there is a bijection $\alpha_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, G(Y) \cong \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ such that for every morphism of objects $\varphi : X \longrightarrow X'$ in $\mathcal{C}$ and $\psi : Y \longrightarrow Y'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \arrow[r, shift left, "\alpha_{X,Y}"] \arrow[d, "{(\varphi, G(\psi))}"'] & \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \arrow[l, shift left] \arrow[d, "{(F(\varphi), \psi)}"] \\
\mathrm{Hom}_{\mathcal{C}}(X', G(Y)) \arrow[r, shift left, "\alpha_{X',Y'}"] & \mathrm{Hom}_{\mathcal{D}}(F(X'), Y') \arrow[l, shift left]
\end{tikzcd}$$

Suppose F and F' are left adjoints to $G : \mathcal{D} \longrightarrow \mathcal{C}$. For X in $\mathcal{C}$ let $\varphi : F(X) \longrightarrow F'(X)$ be the image of $\mathrm{id}_{F'(X)}$ under the adjointness isomorphisms

$$\mathrm{Hom}(F(X), F'(X)) \cong \mathrm{Hom}(X, GF'(X)) \cong \mathrm{Hom}(F'(X), F'(X)).$$

Define $\psi : F'(X) \longrightarrow F(X)$ similarly. It can be verified that φ and ψ are natural isomorphisms. Thus, any two left adjoints of G are naturally isomorphic, the same reasoning gives us that any two right adjoints of G are also naturally isomorphic.

Example. Let A be a commutative ring and let be any A-module. The functor $N \longmapsto M \otimes_A N$ from A-mod to itself is the left adjoint of the functor $N \longmapsto \mathrm{Hom}_A(M, N)$.

Example. Let U be the functor that takes a Lie algebra to its universal enveloping algebra and let L be the functor that takes an associative algebra to its Lie algebra. Then U is the left adjoint to L.

If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a left adjoint to $G : \mathcal{D} \longrightarrow \mathcal{C}$, then for each object Y of $\mathcal{D}$ we have that

$$\mathrm{Hom}_{\mathcal{C}}(F(G(Y)), Y) \cong \mathrm{Hom}_{\mathcal{D}}(G(Y), G(Y)).$$

Let $\varepsilon_Y : F(G(Y)) \longrightarrow Y$ be the image of the identity morphism $G(Y) \longrightarrow G(Y)$, then ε_B is called the *counit*, and similarly for each object X of $\mathcal{C}$ we get a morphism $\eta_X : X \longrightarrow G(F(X))$ called the *unit* of the adjoint pair. The adjointness isomorphism α may be recovered from ε and η. Given a map $\varphi : F(X) \longrightarrow Y$ $G(\varphi) \circ \eta_X$ gives us the corresponding map $X \longrightarrow G(Y)$. Similarly, given a map $\psi : X \longrightarrow G(Y)$ the corresponding map $F(X) \longrightarrow Y$ is given by $\varepsilon \circ F(\psi)$.

7 Direct and Inverse Limits

Let I be a set of indices with a partial ordering. We say that I is *directed* if given $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let $I = \{i\}$ be a directed system of indices. Let $\mathcal{C}$ be a category and $\{X_i\}$ a family of objects in $\mathcal{C}$. For each pair i, j such that $i \leq j$ assume given a morphism

$$f^i_j : X_i \longrightarrow X_j$$

such that whenever $i \leq j \leq k$

$$f^j_k \circ f^i_j = f^i_k$$

and $f^i_i = \mathrm{id}$. Such a family is called a *directed family of morphisms.* A direct limit for the family f^i_j is a universal object in the following category. The objects consist of pairs $(X, (f^i))$ where

$X \in \mathrm{Ob}(\mathcal{C})$ and (f^i) is a family of morphisms $f^i : X_i \longrightarrow X$, such that for all $i \leq j$ the following diagram commutes

$$\begin{tikzcd}
X_i \arrow[dr, "f^i"'] & & X_j \arrow[ll, "f^i_j"'] \arrow[dl, "f^j"] \\
& X &
\end{tikzcd}$$

Thus if $(X, (f^i))$ is the direct limit and if $(X', (g^i))$ is any object in the above category, then there exists a unique morphism $\varphi : X \longrightarrow X'$ making the following diagram commute

$$\begin{tikzcd}
X_i \arrow[rr, "f^i_j"] \arrow[dr, "f^i"] \arrow[ddr, "g^i"'] & & X_j \arrow[dl, "f^j"'] \arrow[ddl, "g^j"] \\
& X \arrow[d, "\varphi"] & \\
& X' &
\end{tikzcd}$$

For simplicity we write $A = \varinjlim A_i$. Reversing the arrows we define inverse limits. Given a directed set I and a family of objects X_i. If $j \geq i$ we are now given a morphism $f^j_i : X_j \longrightarrow X_i$ satisfying $f^i_k \circ f^j_k = f^j_k$ and $f^i_i = \mathrm{id}$, if $j \geq i \geq k$. We now deine a category of objects $(X, (f_i))$ with $f_i : X \longrightarrow X_i$ such that the following diagram commutes

$$\begin{tikzcd}
& X \arrow[dl, "f_j"'] \arrow[dr, "f_i"] & \\
X_j \arrow[rr, "f^j_i"'] & & X_i
\end{tikzcd}$$

A universal object in this category is called an *inverse limit* and we say

$$X = \varprojlim X_i.$$

References

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