Toy example: Graph homology.

Let Γ be a graph with vertices $V=\{v_i\}$ and edges $E=\{e_i\}$. Set $C_0=\mathbb{C}\text{-span}\{v \mid v\in V\}$ and $C_1=\mathbb{C}\text{-span}\{e \mid e\in E\}$.

Fix an orientation of Γ and define $d: C_1\to C_0$ by assigning an (oriented) edge from v_i to v_j to the element v_i-v_j in C_0. This produces a chain complex C:

$$0\longrightarrow C_1 \xrightarrow{d} C_0\longrightarrow 0.$$

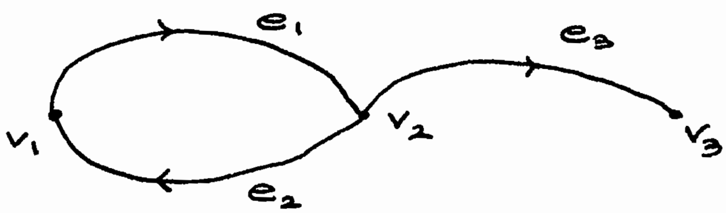
Any basis of $H_1(C)$ is in bijection with minimal circuits in Γ. Similarly, any basis of $H_0(C)$ is in bijection with the connected components of Γ. This shows that $H_*(C)$ is independent of the choice of orientation of Γ. (Another way to see this is that changing orientation induces isomorphisms at the level of complexes).

Additionally, suppose that Γ is finite and connected. Then

$$\dim C_1 = \dim H_1(C) + \dim \operatorname{Im}(d),$$
$$\dim C_0 = \dim \operatorname{Im}(d) + \dim H_0(C).$$

Consequently, $|E| = |C| + |V| - 1$, where $|C|$ is the number of minimal circuits in Γ.

example



$C_0 = \mathbb{C}\text{-span}\{v_1, v_2, v_3\}$, $C_1 = \mathbb{C}\text{-span}\{e_1, e_2, e_3\}$.

$d: C_1 \longrightarrow C_0$,

$e_1 \longmapsto v_1 - v_2$,

$e_2 \longmapsto v_2 - v_1$,

$e_3 \longmapsto v_2 - v_3$.

$H_1(C) = \mathbb{C}\text{-span}\{e_1 - e_2\} \simeq \mathbb{C}$

$H_0(C) = \mathbb{C}\text{-span}\{v_1, v_2, v_3\} \big/ \mathbb{C}\text{-span}\{v_1-v_2, v_2-v_1, v_2-v_3\} \simeq \mathbb{C}$.