Characters traces and fixed points

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A - dualizable object in a symmetric monoidal ∞-category

$$\begin{array}{c} \dim A \in \mathrm{End}(\mathbb{1}) \\ \parallel \\ \mathrm{Tr}\, id_A \end{array}$$

$$\mathrm{Tr}(\varphi\psi) = \mathrm{Tr}(\psi\varphi)$$

$$A \rightsquigarrow \mathcal{C}$$

"object of $\mathcal{C}$"

$\rightarrow$

$$\begin{tikzcd} 1 \arrow[r, "V"] \arrow[dr, "\pi_* V"'] & \mathcal{C} \arrow[d, "\pi_*"] \\ & \mathcal{D} \end{tikzcd}$$

say has right adjoint $\pi^!$

"Formal Grothendieck - Riemann - Roch"

$$\dim(\pi_* V) = (\dim \pi_*)(\dim V)$$

$$\begin{tikzcd} 1 \arrow[r, "\dim V"] \arrow[dr, "\dim(\pi_* V)"'] & \dim \mathcal{C} \arrow[d, "\dim \pi_*"] \\ & \dim \mathcal{D} \end{tikzcd}$$

<u>categorical setting</u>

$\mathcal{C}$ dualizable (e.g. compactly generated)

$\dim \mathcal{C} = HH_* \mathcal{C}$

$V \in \mathcal{C} \rightsquigarrow [V] \in HH_* \mathcal{C}$

Corr = 2-category of correspondences

objects – spaces

1-mor – correspondences

2-mor – maps of correspondences

Any X is dualizable

$$\begin{tikzcd}
 & & \mathcal{L}X \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl] \arrow[dr] & & X \arrow[dl] \arrow[dr] & \\
\bullet & & X \times X & & \bullet
\end{tikzcd}$$

$\dim X = \mathcal{L}X$

$X \xrightarrow{\pi} Y \qquad \dim \pi : \mathcal{L}X \xrightarrow{\mathcal{L}\pi} \mathcal{L}Y$

<u>sheaf theory</u>

$\mathrm{Shv} : \mathrm{Corr} \longrightarrow \mathrm{Cat}$

$X \longrightarrow \mathrm{Sh}(X)$

<u>Examples</u>

$Q^! = \mathrm{QCoh}(X)$ for X smooth
in general $\mathrm{IndCoh}(X)$

$\mathcal{D} = \mathcal{D}\text{-mod}(X)$

$$\dim(\mathrm{Sh}(X)) = \mathrm{Sh}\left(\cdot \leftarrow \mathcal{L}X \rightarrow \cdot\right)$$
$$= \Gamma(\mathcal{O}_{\mathcal{L}X})$$

$\pi : X \longrightarrow Y$ proper

$$\begin{tikzcd} \dim \pi_* : \mathcal{O}(\mathcal{L}X) \arrow[r, "\mathcal{L}\pi_*"', "\underset{\dim \pi_*}{=}"] \arrow[d, equal] & \mathcal{O}(\mathcal{L}Y) \arrow[d, equal] \\ \Omega^{\cdot}_{d}(X) \arrow[r] & \Omega^{\cdot}_{d}(Y) \end{tikzcd}$$

$$\begin{tikzcd} V \in \mathrm{Sh}(X)\ ; & {[V]} \in \dim \mathrm{Sh}(X) \arrow[d] & = & HH_* X \arrow[d, "\dim \pi_*"] \\ & {[\pi_* V]} & \in & HH_* Y \end{tikzcd}$$

$$\begin{tikzcd} G \curvearrowright X, & Z = X/G \arrow[d, "\pi"] & \mathcal{L}(X/G) = \{g \in G,\ x \in X^g\}/G \arrow[d] \arrow[rd, "\mathcal{L}\pi"] & \\ & pt/G & \mathcal{L}(pt/G) & = G/^{ad}G \end{tikzcd}$$

V - G equiv. vector bundle on X

$$\updownarrow$$

$$\mathcal{V} \longrightarrow Z = X/G$$

$\pi_* \mathcal{V}$ = virtual rep of G
$\in \mathrm{vect}(pt/G)$

$\Rightarrow$ Atiyah - Bott - Lefschetz

$[\pi_* V]$ is given by integration over fixed pts.

$\rightsquigarrow$ Weyl character formula

$$
\begin{tikzcd}
X = G/K \arrow[r, squiggly] & Z = G\backslash X = pt/K \arrow[d] & \mathcal{L}(pt/K) = K/_{ad}K \arrow[d] \\
 & pt/G & \mathcal{L}(pt/G) = G/_{ad}G
\end{tikzcd}
$$

$\cancel{(\mathcal{L}(pt/K)) \Rightarrow}$ for $K = B$

$$
\begin{tikzcd}
\mathcal{L}(pt/B) \arrow[d] \\
\mathcal{L}(pt/G)
\end{tikzcd}
$$

is the Grothendieck-Springer resolution.

$$Sh : Corr \longrightarrow 2Cat$$

$$X \longmapsto QCoh(X)\text{-mod}$$

$$\mathcal{D}(X)\text{-mod}$$

$G \circlearrowright X \qquad \mathcal{D}(G) \circlearrowright \mathcal{D}(X)$

$$
\begin{tikzcd}
\mathcal{D}(\mathcal{L}X/G) \arrow[d] & \ni & \mathcal{O} \arrow[d, "\mathcal{L}\pi"] \\
\mathcal{D}(G/_{ad}G) & \ni & \mathcal{L}\pi_* \mathcal{O}
\end{tikzcd}
$$

take $X = G/B$

<u>Thm</u> $\dim_{\mathcal{D}(G)\text{-mod}}\left(\mathcal{D}(G/B)\right) = S$

$\uparrow$

Springer sheaf.

$MC(\mathfrak{g}, K)$-mod

$\wr$ [BB]

$$\mathcal{D}(K\backslash G/B) = \mathcal{D}(G\backslash(G/K \times G/B))$$

$$\| $$

$$\mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{D}(G/B), \mathcal{D}(G/K)) \quad \text{apply } \widehat{\dim}$$

$$\begin{array}{ccc} \dim(\mathcal{D}(G/B)) & \xrightarrow{\dim M} & \dim(\mathcal{D}(G/K)) \\ \| & & \| \\ \delta & \xrightarrow{\chi_M} & \delta_K \quad \text{in } \mathcal{D}(G/_{\mathrm{ad}}G) \\ \| & & \end{array}$$

Harish-Chandra system =: HC

$\rightsquigarrow$ getting sol. of HC-system valued in δ_K

Harish-Chandra wanted characters for admissible $G_{\mathbb{R}}$-reps. (this corresponds to $(\mathfrak{g}, K)$-mods)

$\rightsquigarrow$ well defined as <u>distributions</u> on $G_{\mathbb{R}}$

If V has infinitesimal character $Z \xrightarrow{\lambda} \mathbb{C}$

<u>Geometric Langlands</u>

Σ - alg. curve $/\mathbb{C}$ $\qquad \mathrm{Bun}_G(\Sigma) \ni P$

<u>cartoon</u>:

P trivializes outside some number of pts,

describe P by gluing data $\rightsquigarrow LG = G(\mathbb{C}((z)))$

$G(\mathbb{C}(\Sigma)) \backslash \prod_i' G(\mathbb{C}((z_i))) / G(\mathbb{C}[[z_i]])$

if G is semisimple, suffices to take a single pt. $\rightsquigarrow G(\mathbb{C}(\Sigma - x)) \backslash \underbrace{LG / L^+G}_{\text{affine Grassmanian}}$

$$\begin{array}{ccc} \widehat{\mathrm{Bun}_{G,x}\Sigma} & \hookleftarrow & LG \\ \downarrow & L^+G\text{-torsor} & \\ \mathrm{Bun}_G\Sigma & & \end{array}$$

$$\mathcal{D}(\mathrm{Bun}_G(\Sigma)) \hookleftarrow \mathcal{H}_{sph} = \mathcal{D}(L^+G \backslash LG / L^+G)$$

$I \subset LG$
$\uparrow$
Iwahori

$$\mathcal{H}_{aff} \hookrightarrow \mathcal{D}(\mathrm{Bun}_G\Sigma, \text{flag at } x)$$
$$\shortparallel$$
$$\mathcal{D}(I \backslash LG / I)$$

<u>Problem</u> Decompose $\mathcal{D}(\mathrm{Bun}_G X)$ as module for Hecke symmetries

at the abelian level $\mathcal{H}_{sph}$ is commutative

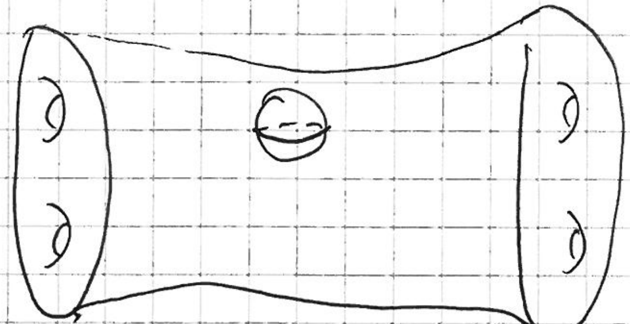
but at the derived level 'correction' is required.

$$\text{“} \operatorname{spec} \bigotimes_{x \in X - \mathrm{ram}} \operatorname{Rep} G^\vee \text{”}$$
$$\text{??}$$
$$\mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\Sigma - \mathrm{ram}))$$

$\rightsquigarrow$ all of this fits in the framework of a 4d TFT (Kapustin-Witten)

$$Z_G(\Sigma) = \mathcal{D}(\mathrm{Bun}_G \Sigma)$$

$$Z_G(S^2) = \mathcal{H}_{sph}$$



$$\Sigma \sqcup S^2 \longrightarrow \Sigma$$

$$\rightsquigarrow Z_G$$

$$\mathcal{D}(\mathrm{Bun}_G \Sigma) \otimes \mathcal{H}_{sph} \longrightarrow \mathcal{D}(\mathrm{Bun}_G \Sigma)$$

$\rightsquigarrow \mathcal{H}_{sph}$ is commutative

Arthur - Selberg trace formula is version of Frobenius character formula

Δ Character of a rep. is a refinement of dim

$$HH_*(\mathcal{H}) = \text{char. sheaves on } G$$

$$HH_*(\mathcal{H}_{aff}) \rightsquigarrow \underbrace{\mathcal{D}_{nil}(LG/_{ad}LG)}_{\text{don't know how to define this}}$$

cheat: $\overset{!}{=}$ character sheaves on LG

But $LG/^{q}LG$ = G-bundles on $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ ~~can be handled~~ is ~~more~~ a close enough substitute

$$\rightsquigarrow HH_*(\mathcal{H}_{aff}) \longrightarrow Z_G(T^2)$$

<u>"Thm"</u> $HH_*(\mathcal{H}_{aff}) \simeq \mathcal{D}_{nil}(\mathrm{Bun}_G E)$

<u>Cor</u> Geometric Langlands version for E

Thanks to [Bezrukavnikov] $\mathcal{H}_{aff} \simeq St_{G^\vee}$