Cobordism hypothesis

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Recall : 1-d TFT : $Z : \mathrm{Bord}\,\mathrm{Cob}_1^{or} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$

0-mnfld $\mapsto$ vector space
1-mnfld $\mapsto$ linear maps

$$Z(pt_-) \simeq Z(pt_+)^*$$

Cobordism hypothesis : Z being determined by $Z(pt_+)$ is quite common, i.e., Cob_n^{fr} is freely generated by a single pt.

$$\mathrm{Fun}^{\otimes}(\mathrm{Cob}_n^{fr}, \mathcal{C}) \longrightarrow (\mathcal{C}^{fd})^{\sim}$$

$$Z \longmapsto Z(pt)$$

Cobordism hypothesis $\rightsquigarrow$ this is an equivalence.

~~2-morphi~~ Informally, $\mathcal{C}^{fd}$ is the subcategory of the (∞, n)-category $\mathcal{C}$ s.t. objects in $\mathcal{C}^{fd}$ have duals, all k-morphisms have adjoints

$$(\infty, n)\text{-category} \rightsquigarrow 2\text{-category}$$

$$\mathcal{C} \longmapsto h_2\mathcal{C}$$

$h_2\mathcal{C}$:

objects = objects of $\mathcal{C}$

1-mor = 1-mor of $\mathcal{C}$

2-mor = 2-mor up to iso { (not equivalence) }

We say $\mathcal{C}$ has adjoints ~~if $h_2\mathcal{C}$~~ for 1-mor if $h_2\mathcal{C}$ does. For $2 \leq k \leq n-1$, we say that $\mathcal{C}$ has adjoints for $(k-1)$-morphisms if the $(\infty, n-1)$-category $\mathrm{Map}_{\mathcal{C}}(X,Y)$ has adjoints for $(k-1)$-morphisms

$O(n) \curvearrowright \mathrm{Bord}_n^{\mathrm{fr}} \rightsquigarrow (\mathcal{C}^{fd})^{\sim}$ has an $O(n)$-action

A G-structure on a manifold M^n is a map $p : G \to O(n)$ and a reduction of the structure group of the tangent frame bundle to G.

Examples

- $G = SO(n)$, a G-str. is an orientation
- $G = e$, a G-str. is an n-framing
- $G = O(n)$, a G-str. is no structure at all.

Remark X^{hG} = homotopy fixed pts.

$= \mathrm{Maps}_G(EG, X)$

$\rightsquigarrow$ Have cobordism hypothesis

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \longrightarrow ((\mathcal{C}^{fd})^{\sim})^{hG}$$