Characters and such

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14 Aug 2012

- Pivotal tensor categories categorify Frobenius algebras!

Main idea for today: Frobenius algebras have really nice representation theory.

A - Frobenius algebra

$$\begin{tikzcd}
 & & A\text{-mod} \arrow[lld] \arrow[rrd] & & \\
 & & M \arrow[ld] \arrow[rd] & & \\
Z(A) \arrow[rrrr, bend right, leftrightarrow, "\sim"'] & Z(A)e_M \arrow[rr, leftrightarrow] & & \chi_M & cl(A)
\end{tikzcd}$$

Note: "tensor" = "monoidal, rigid" + k-linear abelian
or
+ stable ∞, 'complete'

Recall A Frobenius algebra A is an algebra w/ a trace $tr : A \longrightarrow \mathbb{C}$ s.t.

1) $\langle a,b \rangle = tr(ab)$ is non-degenerate

2) $tr(ab) = tr(ba)$.

A tensor category $\mathcal{C}$ is pivotal if given a natural monoidal isomorphism

$$X^{**} \xrightarrow{\sim} X$$

examples
- $Vec(G)$ — open question - are all fusion categories pivotal?
- $Rep(G)$
- $Rep(H)$ — H semi-simple f.d Hopf algebra

Pivotal $\otimes$-categories have a trace:

$$\mathrm{Tr}: \mathcal{C} \longrightarrow \mathrm{Vect},$$
$$X \longmapsto \mathrm{Hom}(\mathbb{1}, X).$$

<u>Frobenius algebras</u>

$$cl(A) = \left(A/[A,A]\right)^* = \{ f(ab) = f(ba) \}$$

<u>Pivotal $\otimes$-categories</u>

$$cl(\mathcal{C}) = \{ F : \mathcal{C} \longrightarrow \mathrm{Vect} \mid F(X \otimes Y) \simeq F(Y \otimes X) + \text{coherence conditions} \}$$

<u>Thm</u> $Z(A) \xrightarrow{\sim} cl(A)$
$z \longmapsto \langle z, - \rangle$

<u>Thm</u> $Z(\mathcal{C}) \xrightarrow{\sim} cl(\mathcal{C})$
$z \longmapsto \mathrm{Hom}(\mathbb{1}, z \otimes -)$

surjectivity $\rightsquigarrow$ representability
injectivity $\rightsquigarrow$ Yoneda lemma

<u>Recall</u> If $\mathcal{M}$ is a $\mathcal{C}$-module category, then we have a monoidal functor:

$$\mathrm{act}: \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M})$$

we also have:

$$Z\mathrm{act}: Z(\mathcal{C}) \longrightarrow \mathrm{End}_{\mathcal{C}}(\mathcal{M})$$

$\mathrm{Tr}: \mathrm{End}_{\mathcal{C}}(\mathcal{M}) \longrightarrow Z(\mathcal{C})$ is the right adjoint to $Z\mathrm{act}$

The <u>central idempotent</u> associated to a module cat. $\mathcal{M}$ is

$$e_{\mathcal{M}} := \mathrm{Tr}(\mathrm{id}_{\mathcal{M}})$$

<u>Aside</u>: ~~s~~ s.s Frob. alg. A

$$\Rightarrow A \simeq \bigoplus \mathrm{Mat}(x_i) \xrightarrow{e_{x_i}} \mathrm{End}(x_i)$$

Thm $e_{\mathcal{M}}$ is a commutative algebra object in $Z(\mathcal{C})$

(As Tr is a right adjoint to a monoidal functo
it is lax-monoidal $\rightsquigarrow$ algebra structure)

Thm The assignment $\mathcal{M} \longmapsto e_{\mathcal{M}}$ is a 2-functor

{ $\mathcal{C}$-module cats. w/ ambi. mod functors; natural transformations } $\longrightarrow$ { comm. algebras in $Z(\mathcal{C})$ w/ bimodules + maps of bimodules }

Construction: $F: \mathcal{M} \longrightarrow \mathcal{N}$, $F^L \simeq F^R: \mathcal{N} \longrightarrow \mathcal{M}$

Consider $F^R F \in \mathrm{End}_{\mathcal{C}}(\mathcal{M})$, $FF^R \in \mathrm{End}_{\mathcal{C}}(\mathcal{N})$

- is a module/$\mathbb{1}_{\mathcal{M}}$ — is a module/$\mathbb{1}_{\mathcal{N}}$
- $\mathbb{1}_{\mathcal{M}} \to F^R F$ is a module map — $FF^R \longrightarrow \mathbb{1}_{\mathcal{N}}$ is a module map.
- $\mathrm{Tr}(F^R F)$ is an $e_{\mathcal{M}}$-module — $\mathrm{Tr}(FF^R)$ is an $e_{\mathcal{N}}$-module

Prop $\mathrm{Tr}(F^R F) \simeq \mathrm{Tr}(FF^R)$ (iso. respects $e_{\mathcal{M}}, e_{\mathcal{N}}$-actions)

proof LHS represents $z \longmapsto \mathrm{Hom}_Z(z, \mathrm{Tr}(F^R F))$

$$= \mathrm{Hom}_{\mathcal{M}}(\mathrm{Zact}(z), F^R F)$$

RHS represents $z \longmapsto \mathrm{Hom}_{\mathcal{N}}(\mathrm{Zact}(z), FF^R)$

$\rightsquigarrow$ ambidextrous ~~now~~ and some symbol juggling gives the result.

$(F: \mathcal{M} \longrightarrow \mathcal{N})$ $\quad e_{\mathcal{M}} - e_{\mathcal{N}}$ $\mathrm{Tr}(F^R F)$

Def $\mathcal{M}$ is <u>small</u> if Tr is conservative

Thm The functor $\mathcal{M} \mapsto e_{\mathcal{M}}$ is conservative on small modules