

characters and such

(b. Jordan)

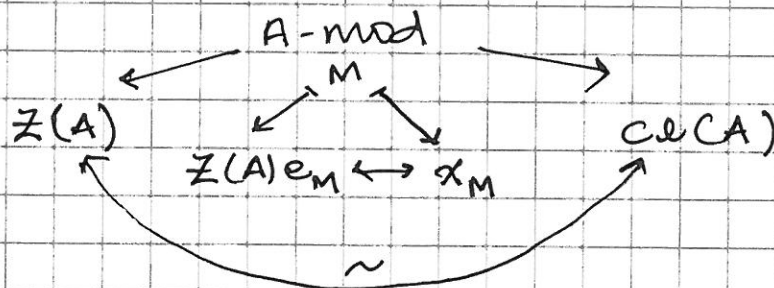
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- Pivotal tensor categories categorify Frobenius algebras!

main idea for today: Frobenius algebras have really nice representation theory.

A - Frobenius algebra



Note: "tensor" = "monoidal, rigid" + K -linear abelian
or
+ stable w, 'complete'

Recall A Frobenius algebra A is an algebra w/ a trace $\text{tr}: A \rightarrow K$ s.t.

1) $\langle a, b \rangle = \text{tr}(ab)$ is non-degenerate

2) $\text{tr}(ab) = \text{tr}(ba)$.

A tensor category $\mathcal{C}$ is pivotal iff given a natural monoidal isomorphism

$$X^{**} \xrightarrow{\sim} X$$

examples

- $\text{Vec}(G)$

- $\text{Rep}(G)$

- $\text{Rep}(H)$

open question - are all fusion categories pivotal?

semi-simple f.d Hopf algebra

Pivotal $\otimes$ -categories have a trace:

(2)

$$\begin{aligned} \text{Tr}: \mathcal{C} &\rightarrow \text{Vect}, \\ x &\mapsto \text{Hom}(\mathbb{1}, x). \end{aligned}$$

Frobenius algebras

$$\text{cl}(A) = \left(A / [A, A] \right)^* = \{ f(ab) = f(ba) \}$$

Thm $\begin{aligned} Z(A) &\xrightarrow{\sim} \text{cl}(A) \\ z &\mapsto \langle z, - \rangle \end{aligned}$

Pivotal $\otimes$ -categories

$$\text{cl}(\mathcal{C}) = \{ F: \mathcal{C} \rightarrow \text{Vect} \}$$

$$F(x \otimes y) \simeq F(y \otimes x) + \text{whence conditions}$$

Thm $\begin{aligned} Z(\mathcal{C}) &\xrightarrow{\sim} \text{cl}(\mathcal{C}) \\ z &\mapsto \text{Hom}(\mathbb{1}, z \otimes -) \end{aligned}$

surjectivity $\leftrightarrow$ representability
injectivity $\leftrightarrow$ Yoneda lemma

Recall If $\mathcal{M}$ is a $\mathcal{C}$ -module category, then we have a monoidal functor:

$$\text{act}: \mathcal{C} \rightarrow \text{End}(\mathcal{M})$$

we also have:

$$Z_{\text{act}}: Z(\mathcal{C}) \rightarrow \text{End}_{\mathcal{C}}(\mathcal{M})$$

$$\text{Tr}: \text{End}_{\mathcal{C}}(\mathcal{M}) \rightarrow Z(\mathcal{C}) \text{ is the}$$

right adjoint to Z_{act}

The central idempotent associated to a module cat.

$\mathcal{M}$ is

$$e_{\mathcal{M}} := \text{Tr}(\text{id}_{\mathcal{M}})$$

Aside: ~~se~~ s.s. Frob. alg. A

$$\Rightarrow A \simeq \bigoplus \text{Mat}(x_i) \xrightarrow{e_{x_i}} \text{End}(x_i)$$

(3)

Thm e_M° is a commutative algebra object in $\mathcal{Z}(e)$

(As Tr° is a right adjoint to a monoidal functor it is lax-monoidal $\leadsto$ algebra structure)

Thm The assignment $M \mapsto e_M^\circ$ is a functor

$\left\{ \begin{array}{l} \text{e-module cats. w/ mod. functors;} \\ \text{natural transformations} \end{array} \right\} \xrightarrow{\text{amb.}} \left\{ \begin{array}{l} \text{comm. algebras in } \mathcal{Z}(e) \\ \text{w/ bimodules + maps of bimodules} \end{array} \right\}$

Construction: $F: M \rightarrow N, F^L = F^R: N \rightarrow M$

Consider $F^R F \in \text{End}_e(M), FF^R \in \text{End}_e(N)$

- $\uparrow$
- $\circ$ is a module / $\mathbb{1}_M$

- $\uparrow$
- $\circ$ is a module / $\mathbb{1}_N$

- $\mathbb{1}_M \rightarrow F^R F$ is a module map

- $FF^R \rightarrow \mathbb{1}_N$ is a module map.

- $\text{Tr}(F^R F)^\circ$ is an e_M -module

- $\text{Tr}(FF^R)^\circ$ is an e_N -module

Prop $\text{Tr}(F^R F) \cong \text{Tr}(FF^R)$ ($\circ$ respects e_M, e_N -actions)

proof LHS represents $z \mapsto \text{Hom}_Z(Z, \text{Tr}(F^R F))$

$\text{Hom}_M(\text{Zact}(Z), F^R F)$

RHS represents $z \mapsto \text{Hom}_N(\text{Zact}(Z), FF^R)$

$\leadsto$ ambidextrous ~~map~~ and some symbol juggling gives the result.

$(F: M \rightarrow N) \quad e_M - e_N \quad \text{Tr}(F^R F)$

Def M° is small if Tr° is conservative

Thm The functor $M \mapsto e_M^\circ$ is conservative on small modules