quasi-coherent sheaves

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k- field, char $k = 0$; X- derived stack w/ affine diagonal

1) three full subcategories of $QCoh(X)$:

$QCoh(X)^c$, $QCoh(X)^d$, $QCoh(X)^p$

Thm $QCoh(X)^d = QCoh(X)^p$

Thm $\mathcal{O}_X$ compact $\Rightarrow QCoh(X)^c = QCoh(X)^d$

2) A perfect stack X is one with $QCoh(X) = Ind\,QCoh(X)^p$

Thm X is perfect $\Leftrightarrow QCoh(X)$ is compactly generated and $QCoh(X)^p = QCoh(X)^c$

example

1) quasi-compact derived schemes

2) Total space of a quasi-projective morphism w/ perfect base

3) X/G, X is quasi-projective $G \curvearrowright X$ is the action of a linear algebraic group

4) X/G X- perfect, G is a finite group scheme

5) $Map(\Sigma, X)$, X- perfect, Σ is a finite simplicial complex

6) Fibre products of perfect stacks

Facts about stacks

Alg = commutative dga $/k$

$Aff = Alg^{op}$

~~Stak~~ stack $\approx$ presheaf (Aff) + descent properties

$QCoh(\operatorname{Spec} A) :=$ ~~Mod~~ A-mod

$$QCoh(X) := \lim_{\operatorname{Spec} A \to X} QCoh(\operatorname{Spec} A)$$

compact objects

$\mathcal{C}$ - stable ∞ - category

$M \in \mathcal{C}$ is compact if $Hom_{\mathcal{C}}(M, -)$ preserves colimits

Thm Any map from a compact object to arbitrary colimit must factor through a finite colimit

Let $\mathcal{C}$ be a closed symmetric monoidal stable ∞-cat.

An object $M \in \mathcal{C}$ is dualizable if

$$\underline{Hom}(M, \mathbb{1}) \otimes - \xrightarrow{\sim} \underline{Hom}(M, -)$$

$\mathcal{C}^d$ = full subcategory of dualizable objects

Thm $\mathcal{C}^d$ is closed under retracts.

Thm If $\mathbb{1}$ is compact, then dualizable $\Rightarrow$ compact

Perfect objects

If $X = \operatorname{Spec} A$, we say $M \in QCoh(\operatorname{Spec} A)$ is perfect if M is contained in the smallest subcategory of A-mod containing A that is closed under retracts and finite colimits.

we say that $M \in QCoh(X)$ is perfect if f^*M is perfect for all $f: \operatorname{spec} A \longrightarrow X$.

<u>Thm</u> $QCoh(\operatorname{spec} A)^d = QCoh(\operatorname{Spec} A)^P = QCoh(\operatorname{Spec} A)^c$

<u>proof</u> dualizable $\Rightarrow$ compact because A is compact

compact $\Rightarrow$ perfect because $M = \operatorname{colim} D$. If M is compact ~~$\cancel{\ast}$~~ $1_M : M \longrightarrow M$ factors through a finite colimit.

perfect $\Rightarrow$ dualizable because $(A[n])^\vee = A[-n]$

$\rightsquigarrow$ finite coproducts of dualizables are dualizables

$\rightsquigarrow$ cofibers of dualizables are dualizable

<u>Thm</u> $QC(X)^d = QC(X)^P$

A stable ∞-category is compactly generated if there exists a set S of compact objects such that for each $M \in \mathcal{C}$: $M = 0 \iff$ ~~hom~~ $\operatorname{Hom}(S, M) = 0$ for all $s \in S$.

<u>Thm</u> If $\mathcal{C}$ is compactly generated, then

$$\mathcal{C} \xrightarrow{\sim} \operatorname{Ind} \mathcal{C}^0$$

$\uparrow$ full subcategory spanned by S.

<u>Thm</u> If $\mathcal{C} = \operatorname{Ind} \mathcal{C}^0$ small, closed under finite colimits, then $\mathcal{C}$ is cocomplete, compactly generated by $\operatorname{Ob} \mathcal{C}^0$; $\mathcal{C}^c$ = idempotent completion of $\mathcal{C}^0$.

<u>Thm</u> X is perfect, $\mathrm{QCoh}(X) = \mathrm{Ind}\,\mathrm{QCoh}(X)^{p}$

$$\Updownarrow$$

$\mathrm{QCoh}(X)$ is compactly generated and $\mathrm{QCoh}(X)^{c} = \mathrm{QCoh}(X)^{p}$