Character theory for surfaces

S. Gunningham

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G - finite group

2d-TFT : Frobenius algebra

$$Z_G(pt) = \mathbb{C}[G]$$

Another approach

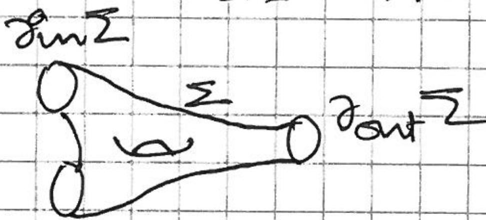
M - manifold (connected)

$$\rightsquigarrow \mathcal{F}(M) = Loc_G(M) = Hom(\pi_1(M), G)/G$$
$$= \coprod_{[P]} pt/Aut(P)$$

Σ - surface

$$Z_G(\Sigma) = Vol(\mathcal{F}(\Sigma))$$
$$= \sum_{[P]} 1/|Aut(P)|$$

If Σ has boundary



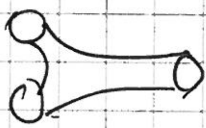
$$\begin{array}{ccc} & \mathcal{F}(\Sigma) & \\ {}^{p}\swarrow & & \searrow^{q} \\ \cancel{\mathcal{F}(\partial\Sigma)} & & \mathcal{F}(\partial_{out}\Sigma) \\ \mathcal{F}(\partial_{in}\Sigma) & & \end{array}$$

$$\rightsquigarrow \mathbb{C}[\mathcal{F}(\partial_{in})] \xrightarrow{q_* p^*} \mathbb{C}[\mathcal{F}(\partial_{out})]$$

Σ'

glue w/ Σ $\rightsquigarrow$ assoc. for integral transforms (base change)

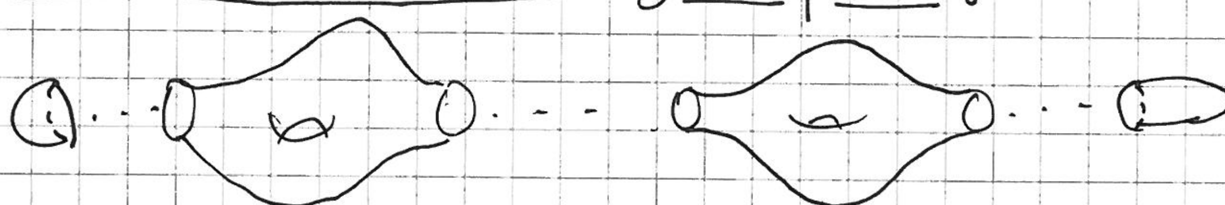
$\rightsquigarrow$ Z_G is a functor

If $\Sigma =$ 

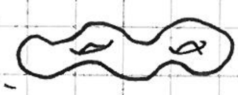
$$\begin{tikzcd} & G\backslash(G\times G) \arrow[dl] \arrow[dr, "\mathrm{mult}"] & \\ G/^{ad}G \times G/^{ad}G & & G/^{ad}G \end{tikzcd}$$

$\rightsquigarrow$ $\mathbb{C}[G/^{ad}G] = Z_G(S^1)$

<u>compute $\mathrm{vol}(\mathcal{F}(\Sigma))$ using rep. thry</u>



$$\begin{tikzcd} & \mathcal{F}(\Sigma) \arrow[dl] \arrow[dr] & \\ pt & & pt \end{tikzcd}$$

$\rightsquigarrow$ $Z_G($  $) = \sum_{V\in\hat{G}} \left(\frac{\dim V}{|G|}\right)^{2-2g}$

$\Sigma^* = \Sigma - pt$

$$\mathcal{F}(\Sigma^*) \xrightarrow{f} \mathcal{F}(S^1) = G/^{ad}G$$

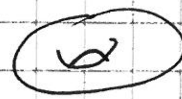
$\langle \delta_1, f_* 1_{\mathcal{F}(\Sigma^*)} \rangle = Z_G(\Sigma)$

for each (irred) $V \in \hat{G}$:

$$\begin{tikzcd} & & \text{Frob. alg.} \\ \mathrm{End}(V) \arrow[urr] \arrow[r] \arrow[d, squiggly] & \mathbb{C}[G] \arrow[r] & \mathbb{C}[G/^{ad}G] \\ Z_{G,V}(\Sigma) = \langle e_V, f_* 1 \rangle & & \end{tikzcd}$$

G - red. group$/\mathbb{C}$ $\quad (SL_2)$

$$\mathcal{F}(M) = Loc_G(M) = \underbrace{Hom(\pi_1(M), G)}_{\text{affine variety}} /\!/ G$$

example $\quad \Sigma = S^1 \times S^1$ 

$$\mathcal{F}(\Sigma) = \{(g_1, g_2) \mid [g_1, g_2] = e\}$$

morally $\quad Z_G(\Sigma) = H^*(\mathcal{F}(\Sigma))$

$$\begin{tikzcd} & \mathcal{F}(\Sigma) \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\ pt & & pt \end{tikzcd}$$

$Z_G(S^1) = \mathcal{D}(G/_{ad}G)$ | $\mathcal{D}(G)$ is not fully dualizable

$Z_G(pt) = \mathcal{D}(G)$-mods | but $\mathcal{D}(B\backslash G/B)$ is.

$$\mathcal{D}(B\backslash G/B) = \mathcal{H} = End_{\mathcal{D}(G)}(\mathcal{D}(G/B))$$

$$\mathcal{F}(\Sigma^*) \longrightarrow G/_{ad}G$$

$V = \mathcal{D}(G/B$. What is $e_\emptyset$?

unit object in Ch

$$\begin{tikzcd} & G/B \arrow[dl] \arrow[dr] & \\ B\backslash G/B & & G/_{ad}G \end{tikzcd}$$

$$\begin{tikzcd} F: \mathcal{H} \arrow[r, shift left] & \mathcal{D}(G/_{ad}G) \arrow[l, dotted, shift left, "F^r"] \end{tikzcd}$$

$$1 \longmapsto S$$

<u>claim</u> $\quad e_0 = \operatorname{colim}\left(S \leftleftarrows S * S \leftleftarrows \cdots \right)$

note: $FF^r M = S * M$

<u>Thm</u> $\quad Z_{\mathcal{H}}(\Sigma) = \langle e_0, f_* \mathcal{O}_{\mathcal{F}(\Sigma^*)} \rangle$

<u>monodromic parameter</u>

$$\text{for each } \bar{\lambda} \in \mathfrak{h}^*/W_{aff} \rightsquigarrow \begin{array}{c} Ch_{\bar{\lambda}} \subseteq \mathcal{D}(G/_{ad}G) \\ \text{\rotatebox{90}{$\in$}} \\ e_\lambda \end{array}$$

"<u>Thm</u>" There is a W_{aff}-equivariant quasi-coh sheaf on $\mathfrak{h}^*$ whose stalks at pts λ are $Z_{G,\lambda}(\Sigma)$ and

$$(\text{global sections})^{sgn} = H^*(\mathcal{F}(\Sigma))^{poinc}$$
$$\downarrow$$
$$H^*(\mathcal{F}(\Sigma))$$