

Representations of finite groups

(1. Gpner)

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13 Aug 2012

- functions on finite sets
- group algebra
- induced reps.
- Hecke algebra
- characters

X - finite set

$$\mathbb{C}[X] = \{ f: X \rightarrow \mathbb{C} \}$$

Algebra structure: $(\mathbb{C}[X], \text{pt. wise multiplication})$

A $\mathbb{C}[X]$ -module is tautologically a vector bundle on X .

$$\begin{array}{ccccc} & & X \times X \times X & & \\ \pi_{12} \swarrow & & \downarrow \pi_{13} & \searrow & \pi_{23} \\ X \times X & & X \times X & & X \times X \end{array}$$

for $A, B \in \mathbb{C}[X \times X]$, define

$$A * B = \pi_{13}^* (\pi_{12}^* A \cdot \pi_{23}^* B)$$

$$\leadsto (\mathbb{C}[X \times X], *) \simeq \text{Mat}_n(\mathbb{C}) \quad n = \#X$$

prop $(\mathbb{C}[X \times X], *)$ is Morita equiv. to $\mathbb{C}$.

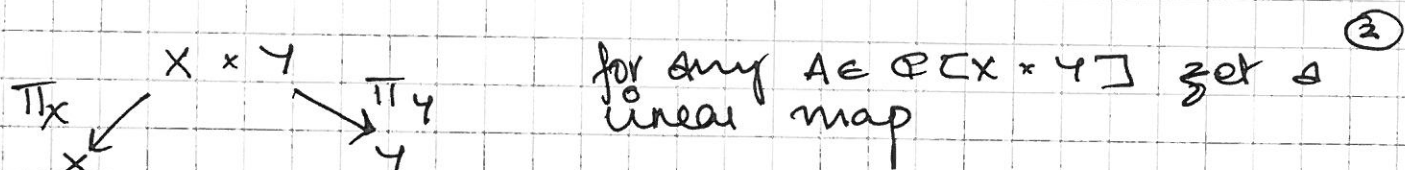
proof sketch Assume $X = \{1, \dots, n\}$ and let V be a $(\mathbb{C}[X \times X], *)$ -module

$$\begin{array}{ccc} (\mathbb{C}[X], \text{pt. wise}) & \longrightarrow & (\mathbb{C}[X \times X], *) \\ \mathbb{Z}_{ii} & \longmapsto & \mathbb{Z}_{ii} = E_{ii} \end{array}$$

$$\leadsto V = \bigoplus_{i=1}^n V_i^0 \quad \leadsto E_{ij}^0 \in \mathbb{C}[X \times X] \text{ gives an iso } V_i^0 \xrightarrow{\sim} V_j^0.$$

Let $\alpha: X \rightarrow Y$ be a surjective map of sets

prop $(\mathbb{C}[X \times_Y X], *)$ is Morita equiv. to $(\mathbb{C}[Y], \text{pt. wise})$

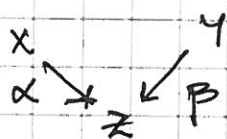


for any $A \in \mathcal{A}[X \times Y]$ get a linear map

$$\begin{aligned}
 \mathcal{A}[X] &\longrightarrow \mathcal{A}[Y] \\
 f &\longmapsto \pi_{Y*} (\pi_X^* f) \cdot A
 \end{aligned}$$

Prop 1) $\mathcal{A}[X \times Y] \xrightarrow{\sim} \text{Hom}(\mathcal{A}[X], \mathcal{A}[Y])$

2) if $X=Y$, then $(\mathcal{A}[X \times X], *) \xrightarrow{\sim} \text{End}(\mathcal{A}[X])$

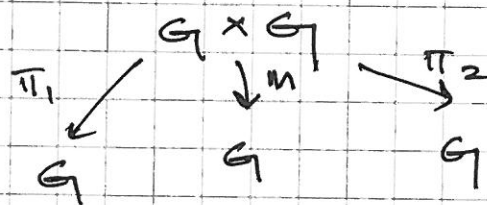


- $\mathcal{A}[X], \mathcal{A}[Y]$ are $\mathcal{A}[Z]$ -modules

- $\mathcal{A}[X \times_Z Y] \xrightarrow{\sim} \text{Hom}_{\mathcal{A}[Z]}(\mathcal{A}[X], \mathcal{A}[Y])$

Group algebra

G -finite group



The group algebra of G is $\mathcal{A}[G]$ w/ multiplication

$$f * g = m_* (\pi_1^* f \cdot \pi_2^* g)$$

note - $\sum g_i * \sum h_j = \sum g_i h_j$

- for any vector space V $\left\{ \begin{smallmatrix} \text{uops of } G \\ \text{in } V \end{smallmatrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{smallmatrix} \mathcal{A}[G]\text{-mod} \\ \text{w/ structure} \end{smallmatrix} \right\}$

$G \mapsto \mathcal{A}[G \times G]$ diagonally

Prop 1) $\mathcal{A}[G \times G]^G$ is a subalgebra of $\mathcal{A}[G \times G]$

2) $(\mathcal{A}[G], *) \simeq (\mathcal{A}[G \times G]^G, \text{matrix mult})$

The class functions on G

$$\mathcal{A}[G / G^{\text{conj}}] = \left\{ f \in \mathcal{A}[G] \mid f(n g n^{-1}) = f(g), \forall g \in G, \forall n \in G \right\}$$

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$$- \mathbb{C}[G/\overset{\circ}{G}] = \mathbb{Z}[\mathbb{C}[G]]$$

$$- \mathbb{C}[G/\overset{\circ}{G}] = \mathbb{C}[G] / a \# b - b \# a$$

Induced reps let K be a subgroup of G , then $\mathbb{C}[K]$ is a subalgebra of $\mathbb{C}[G]$

$$\text{Res: Rep } G \rightarrow \text{Rep } K$$

$$\text{left adjoint} \quad \text{Ind: Rep } K \rightarrow \text{Rep } G,$$

$$W \mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W$$

$$\text{right adjoint} \quad \text{coInd: Rep } K \rightarrow \text{Rep } G,$$

$$W \mapsto \text{Hom}_G(\mathbb{C}[G], W).$$

As we are in the finite group setting $\text{Ind} \simeq \text{coInd}$

$$\text{Hecke algebras} \quad K \leq G, \quad V \in \text{Rep}(G)$$

$$V^K = \{ v \in V \mid kv = v, \forall k \in K \}.$$

This functor is representable by $\mathbb{C}[G/K]$

$$\leadsto \text{So } \text{End}_G(\mathbb{C}[G/K]) \xrightarrow{\sim} V^K$$

The Hecke algebra of the pair (G, K) is

$$\mathcal{H}(G, K) = \text{End}_G(\mathbb{C}[G/K]).$$

Note:

$$- \text{Yoneda lemma} \Rightarrow \mathcal{H}(G, K)^{\text{op}} = \text{End}(-)^K$$

$$- \text{If } G \twoheadrightarrow X, \text{ then } \mathbb{C}[X] \text{ is a rep. of } G, \text{ and } \mathcal{H}(G, K) \twoheadrightarrow \mathbb{C}[X]^K = \mathbb{C}[K \backslash X]$$

- By Bruin-Beck:

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$$\mathcal{H}(G, K) \text{ - mod } \cong \langle \mathcal{C}[G/K] \rangle$$

= full subcat. of $\text{Rep } G$ consisting
 of 0 and V s.t.
 $\text{Hom}_G(\mathcal{C}[G/K], V) \neq 0$

Prop $K \hookrightarrow \mathcal{C}[G] \hookleftarrow K \rightsquigarrow \mathcal{C}[K \backslash G/K]$ is a
 subalgebra of $\mathcal{C}[G]$ isomorphic to $\mathcal{H}(G, K)$

$$\text{Also } \mathcal{H}(G, K) \xrightarrow{\sim} \mathcal{C}[G/K \times G/K]^G$$

Characters

Let V be a finite dim. rep. of G

$$\begin{aligned} \phi: \text{End}_G(V) &\cong V^* \otimes V \longrightarrow \mathcal{C}[G] \\ v^* \otimes v &\longmapsto g \longmapsto \langle v^*, g \cdot v \rangle \end{aligned}$$

The character of G on V is $\chi_V = \phi(\text{id}_V)$

- If $G \curvearrowright X$, then $\chi_{\mathcal{C}[X]}(g) = \# \{x \in X \mid gx = x\}$

- $\chi_V \in \mathcal{C}[G/G]$

- Let $\chi_1, \dots, \chi_n$ be irred. reps of G (complete set of iso classes)

then the corresponding characters $\chi_1, \dots, \chi_n$ form a
 basis of $\mathcal{C}[G/G]$

- $\chi_{V \oplus W} = \chi_V + \chi_W$; $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

- $K_0(\text{Rep } G) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{C}[G/G]$

K subgroup of G ; $\pi: K/K \rightarrow G/G$

χ_w be the character of $w \in K$ -rep

Prop (Frobenius character formula)

$$\chi_{\text{ind } W} = \overline{\pi}_* (\psi_W) = \frac{1}{|K|} \sum_{n \in G} \psi_W(n g n^{-1})$$