Representations of finite groups

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- functions on finite sets
- group algebra
- induced reps.
- Hecke algebra
- characters

X - finite set $\qquad \mathbb{C}[X] = \{ f : X \longrightarrow \mathbb{C} \}$

Algebra structure : $(\mathbb{C}[X]$, pt. wise multiplication$)$

A $\mathbb{C}[X]$ - module is tautologically a vector bundle on X.

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ X \times X & X \times X & X \times X \end{tikzcd}$$

for $A, B \in \mathbb{C}[X \times X]$, define

$$A * B = \pi_{13*}(\pi_{12}^* A \cdot \pi_{23}^* B)$$

$\rightsquigarrow (\mathbb{C}[X \times X], *) \simeq \mathrm{Mat}_n(\mathbb{C}) \qquad n = \# X$

prop $(\mathbb{C}[X \times X], *)$ is Morita equiv. to $\mathbb{C}$.

proof sketch Assume $X = \{1, \dots, n\}$ and let V be a $(\mathbb{C}[X \times X], *)$ - module

$$(\mathbb{C}[X], \text{pt. wise}) \longrightarrow (\mathbb{C}[X \times X], *)$$
$$\delta_i \longmapsto \delta_{ii} = E_{ii}$$

$\rightsquigarrow V = \bigoplus_{i=1}^{n} V_i . \rightsquigarrow E_{ij} \in \mathbb{C}[X \times X]$ gives an iso $V_i \xrightarrow{\sim} V_j$. $\square$

Let $\alpha : X \longrightarrow Y$ be a surjective map of sets

prop $(\mathbb{C}[X \times_Y X], *)$ is Morita equiv. to $(\mathbb{C}[Y]$, pt. wise$)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

for any $A \in \mathbb{C}[X \times Y]$ get a linear map

$$\mathbb{C}[X] \longrightarrow \mathbb{C}[Y]$$
$$f \longmapsto \pi_{Y*}(\pi_X^* f \cdot A)$$

Prop 1) $\mathbb{C}[X \times Y] \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}[X], \mathbb{C}[Y])$

2) If $X = Y$, then $(\mathbb{C}[X \times X], *) \xrightarrow{\sim} \mathrm{End}(\mathbb{C}[X])$

$$\begin{tikzcd} X \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\ & Z & \end{tikzcd}$$

- $\mathbb{C}[X]$, $\mathbb{C}[Y]$ are $\mathbb{C}[Z]$-modules
- $\mathbb{C}[X \times_Z Y] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}[Z]}(\mathbb{C}[X], \mathbb{C}[Y])$

Group algebra G - finite group

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[d, "m"] \arrow[dr, "\pi_2"] & \\ G & G & G \end{tikzcd}$$

The group algebra of G is $\mathbb{C}[G]$ w/ multiplication

$$f * g = m_*(\pi_1^* f \cdot \pi_2^* g)$$

note
- $\delta_g * \delta_h = \delta_{gh}$
- for any vector space V $\left\{ \begin{array}{c} \text{reps of } G \\ \text{in } V \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \mathbb{C}[G]\text{-mod} \\ \text{ule struct.} \\ \text{on } V \end{array} \right\}$

$G \curvearrowright \mathbb{C}[G \times G]$ diagonally

Prop 1) $\mathbb{C}[G \times G]^G$ is a subalgebra of $\mathbb{C}[G \times G]$

2) $(\mathbb{C}[G], *) \simeq (\mathbb{C}[G \times G]^G, \text{matrix mult})$

The class functions on G

$$\mathbb{C}[G/_{G}^{\text{conj}}] = \{ f \in \mathbb{C}[G] \mid f(xgx^{-1}) = f(g), \ \forall g \in G, \ \forall x \in G \}$$

- $\mathbb{C}[G/_{G}^{\text{conj}}] = Z(\mathbb{C}[G])$
- $\mathbb{C}[G/_{G}^{\text{conj}}] = \mathbb{C}[G]/_{a*b-b*a}$

<u>Induced reps</u> Let K be a subgroup of G, then $\mathbb{C}[K]$ is a subalgebra of $\mathbb{C}[G]$

$$\mathrm{Res}: \mathrm{Rep}\, G \longrightarrow \mathrm{Rep}\, K$$

<u>left adjoint</u> $\mathrm{Ind}: \mathrm{Rep}\, K \longrightarrow \mathrm{Rep}\, G,$

$$W \longmapsto \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W$$

<u>right adjoint</u> $\mathrm{coInd}: \mathrm{Rep}\, K \longrightarrow \mathrm{Rep}\, G,$

$$W \longmapsto \mathrm{Hom}_{\mathbb{C}[K]}(\mathbb{C}[G], W).$$

As we are in the finite group setting $\mathrm{Ind} \simeq \mathrm{coInd}$

<u>Hecke algebras</u> $K \leq G$. $V \in \mathrm{Rep}(G)$

$V^K = \{ v \in V \mid kv = v, \forall k \in K \}$.

This functor is representable by $\mathbb{C}[G/K]$

$\rightsquigarrow$ So $\mathrm{End}_G(\mathbb{C}[G/K]) \curvearrowright V^K$

The <u>Hecke algebra</u> of the pair (G, K) is

$$\mathcal{H}(G,K) = \mathrm{End}_G(\mathbb{C}[G/K]).$$

<u>Note</u>:

- Yoneda lemma $\Rightarrow \mathcal{H}(G,K)^{op} = \mathrm{End}(-)^K$
- If $G \curvearrowright X$, then $\mathbb{C}[X]$ is a rep. of G, and
  $\mathcal{H}(G,K) \curvearrowright \mathbb{C}[X]^K = \mathbb{C}[K\backslash X]$
- By Barr-Beck:

$\mathcal{H}(G,K)$-mod $\simeq \langle \mathbb{C}[G/K] \rangle$

= full subcat. of Rep G consisting of 0 and V s.t. $\mathrm{Hom}_G(\mathbb{C}[G/K], V) \neq 0$

<u>Prop</u> $K \curvearrowright \mathbb{C}[G] \curvearrowleft K \rightsquigarrow \mathbb{C}[K\backslash G/K]$ is a subalgebra of $\mathbb{C}[G]$ isomorphic to $\mathcal{H}(G,K)$

Also $\mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[G/K \times G/K]^G$

<u>Characters</u>

Let V be a finite dim. rep. of G

$$\phi : \mathrm{End}_{\mathbb{C}}(V) \simeq V^* \otimes V \longrightarrow \mathbb{C}[G]$$
$$v^* \otimes v \longmapsto g \longmapsto \langle v^*, g \cdot v \rangle$$

The <u>character</u> of G on V is $\chi_V = \phi(\mathrm{id}_V)$

- If $G \curvearrowright X$, then $\chi_{\mathbb{C}[X]}(g) = \#\{x \in X \mid gx = x\}$
- $\chi_V \in \mathbb{C}[G/_{G}^{\mathrm{conj}}]$
- Let $L_1, \dots, L_n$ be irred. reps of G (complete set of iso classes) then the corresponding characters $\chi_1, \dots, \chi_n$ form a basis of $\mathbb{C}[G/_{G}^{\mathrm{conj}}]$
- $\chi_{V \oplus W} = \chi_V + \chi_W$; $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
- $K_0(\mathrm{Rep}\, G) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[G/_{G}^{\mathrm{conj}}]$.

K subgrp of G ; $\pi : K/_{K}^{\mathrm{conj}} \longrightarrow G/_{G}^{\mathrm{conj}}$

ψ_W be the character of $W \in K$-rep

<u>Prop</u> (Frobenius character formula)

$$\chi_{\mathrm{Ind}\, W} = \pi_*(\chi_W) = \frac{1}{|K|} \sum_{x \in G} \chi_W(xgx^{-1})$$