TQFTs

#1 (Z. Dancso ?, J. Thind)
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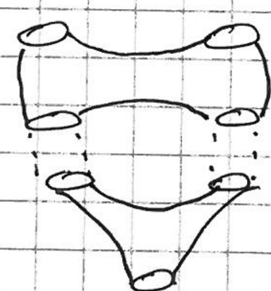
- An n-dim TQFT is a symmetric monoidal functor $Z: \mathrm{Cob}(n) \longrightarrow \mathrm{Vect}$
- $\mathrm{Cob}(n)$: objects - closed, oriented $(n-1)$ manifolds
  morphisms - cobordisms $M \xrightarrow{B} N$, B is an n-manifold w/ orientation preserving diffeomorphism $\partial B \longrightarrow \overline{M} \sqcup N$ up to orientation preserving diffeomorphisms

composition of cobordisms given by gluing



$\otimes$ is disjoint union ; ϕ is unit

If B^n is a closed manifold $Z:$ (torus) $\longrightarrow$ $k \uparrow k$ (a number)

Examples $n=1$

0-manifolds, i.e., finite collections of oriented pts.

$\overset{+}{\cdot}$ $\overset{-}{\cdot}$; ~~$Z(\cdot) = V$~~ $Z(\overset{+}{\cdot}) = V$; $Z(\overset{-}{\cdot}) = W$

$[0,1]$ - 1-manifold

$\rightsquigarrow$ $V^* \cong W$ and V is finite dim

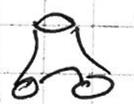
$n=2$

In $\mathrm{Cob}(2)$: objects - closed 1-manifolds, i.e., finite disjoint unions of S^1

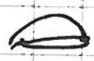
$\rightsquigarrow Z(M) = Z(\sqcup S^1) = \otimes Z(S^1)$.

$\rightsquigarrow Z(S^1)$ completely determines Z on objects.

Recall : Any surface Σ can be obtained by gluing cups :  ; caps :  ; pants : , 

$Z($  $) : Z(S^1) \otimes Z(S^1) \xrightarrow{m} Z(S^1)$

$Z($  $) : Z(S^1) \xrightarrow{\Delta} Z(S^1) \otimes Z(S^1)$

$Z($  $) : Z(S^1) \xrightarrow{\varepsilon} k$

$Z($  $) : k \xrightarrow{\eta} Z(S^1)$

$\rightsquigarrow Z(S^1)$ has structure of a commutative Frobenius algebra.

Thm 1-1 correspondence b/w 2d TFT and commutative Frobenius algebras.

$$Z(S^1) \longleftrightarrow A$$

Extended TFT/TQFT

2-extended want Z to assign something to M^{n-2}.

$\mathrm{Cob}_2(n)$: objects - $n-2$ manifolds
1-morphisms - cobordisms b/w $n-2$ manifolds i.e $n-1$ manifolds
2-morphisms - cobordisms b/w cobordisms i.e. n-manifolds w/ corners

Roughly a 2-extended TQFT is a symm. monoid -al $Z : Cob_2(n) \longrightarrow$ 2-category

One target 2-cat is $Alg_{\mathbb{C}}$: objects - $\mathbb{C}$-alg

1-mor: A-B bimodules ; 2-mor: bimodule maps

<u>Thm</u> 2D-extended TQFTs $\xleftrightarrow{1-1}$ Frobenius algebras (f.dim + semi-simple)

$$Z(\cdot) \longleftrightarrow A$$

$$Z(S^1) = A/_{[A,A]}$$

(look at C. Schommer-Pries thesis).

<u>Cor</u> for (f. dim, s.s) Frob. algebra, center is isomo-rphic to abelianization.

<u>example</u> Γ = finite group

$Z_\Gamma(\cdot) = \mathbb{C}[\Gamma]$ $\quad$ $\cancel{Z_\Gamma(\bigcirc)} = Z_\Gamma(S^1) = \mathbb{C}[\Gamma]^\Gamma$

$Z_\Gamma(\Sigma) = \#\, \mathrm{Hom}(\pi_1(\Sigma), \Gamma)/_\Gamma = \#\{$ pts in space of Γ-bundles on $\Sigma \}$

Σ: surface w/o boundary