V. Ginzburg — W-action on base affine space

We don't really know what G/U is.

Flag variety $= G/B$

Better (or more fashionable)

Flag variety $= \mathcal{B} =$ variety of all Borel subalgebras of Lie G

$G/U = ?$ (can only do it for G adjoint, assume so from now)

whatever G/U is it should map to $\mathcal{B}$ and should be a T-torsor.

$T =$ universal torus $= B/[B,B]$.

unique up to canonical isomorphism.

Pick $\mathfrak{b} \in \mathcal{B}$. $\mathfrak{g} = \mathrm{Lie}\, G$

$$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{u} = [\mathfrak{b},\mathfrak{b}] \supset [\mathfrak{u},\mathfrak{u}]$$

$$\left.\mathfrak{b} \oplus \bigoplus_{\alpha \text{ simple}} \mathbb{C} e_\alpha\right\} \mathfrak{b}' \qquad \mathfrak{u}/[\mathfrak{u},\mathfrak{u}] = \bigoplus_{\alpha \text{ simple}} \mathbb{C} f_\alpha$$

$$\mathfrak{u}/[\mathfrak{u},\mathfrak{u}] \cup \mathbb{O}_+(\mathfrak{b}) \qquad \mathfrak{b}'/\mathfrak{b} \cup \mathbb{O}_-(\mathfrak{b})$$

$$\tilde{\mathcal{B}}_+ = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B},\ x \in \mathbb{O}_+(\mathfrak{b})\} \longrightarrow \mathcal{B}$$
$$(\mathfrak{b}, x) \longmapsto \mathfrak{b}$$

$$\tilde{\mathcal{B}}_- = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B},\ x \in \mathbb{O}_-(\mathfrak{b})\} \longrightarrow \mathcal{B}$$
$$(\mathfrak{b}, x) \longmapsto \mathcal{B}$$

$$\hat{\mathcal{B}}_+ \simeq G/U \simeq \tilde{\mathcal{B}}_-$$

Remark G/U is particularly problematic for G as group scheme/arbitrary ring

<u>definition</u> $x \in \mathfrak{g}$ and $\mathfrak{b} \subset \mathfrak{g}$ are in negative position if $x \in \mathfrak{b} \oplus \bigoplus_{\substack{\alpha \\ \text{simple}}} \mathbb{C} e_{\alpha}$ and ($x \bmod \mathfrak{b} \in \mathcal{O}_-(\mathfrak{b})$)

given $x \in \mathfrak{g}$, $\mathcal{B}_+^x = \{\mathfrak{b} \in \mathcal{B} \mid \mathfrak{b} \ni x\}$. (usual Springer fibre)

$\mathcal{B}_-^x = \{\mathfrak{b} \in \mathcal{B} \mid x$ and $\mathfrak{b}$ are in -ve position$\}$

<u>Lemma</u> $\mathcal{B}_-^x \neq \phi$ iff x is regular

<u>Lemma</u> $\mathcal{B}_-^x$ is a principal homogeneous space for G^x (= centralizer of x in G)

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid \mathfrak{b} \in \mathcal{B}_+^x\}$$

$$X = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid \mathfrak{b} \in \mathcal{B}_-^x\}$$

According to $X \subset \mathfrak{g}^r \times \mathcal{B}$; $\mathfrak{g}^r$ = regular elements in $\mathfrak{g}$

$$\mathcal{G} = \{(x, g) \in \mathfrak{g}^r \times G \mid \mathrm{Ad}\, g(x) = x\}$$

$\mathcal{G} \xrightarrow{\text{project}} \mathfrak{g}^r$ (this is a $\underbrace{\text{group}}_{\text{why?}}$ scheme$/\mathfrak{g}$)

$X \xrightarrow{\text{project}} \mathfrak{g}^r$

<u>Summary</u> $X \to \mathfrak{g}^r$ is a $\mathcal{G}$-torsor on $\mathfrak{g}^r$

$\overline{X}$:= closure of X in $\mathfrak{g}^r \times \mathcal{B}$

$\overline{X} \xrightarrow{\mu} \mathfrak{g}^r$ proper

$\overline{X}$ is smooth

look at projection to $\mathcal{B}$

For x regular semi-simple G^x is a maximal torus and $\overline{\mathcal{B}}_-^x$ is a toric variety associated to Weyl chambers

$\overline{\mathcal{B}}_-^x$

$\bigcup_x$

$\mathcal{B}_+$

torus fixed pts.

if x regular but not s.s, $\overline{\mathcal{B}}_-^x$ = closure of G^x-orbit

<u>open problem</u> does G^x act on $\overline{\mathcal{B}}_-^x$ w/ finitely many orbits?

$$\begin{tikzcd} \overline{X} \arrow[d, "\mu"'] \\ \mathfrak{g}^r \end{tikzcd}$$

certainly not smooth

Analogue of Ngo's support thm:

<u>conjecture</u> $R\mu_* \mathbb{C}_{\overline{X}} = IC(R\mu_* \mathbb{C}_{\overline{X}}|_{\mathfrak{g}^{rs}})$

'negative steinberg variety'

$$Z = \{(x, \mathfrak{b}_+, \mathfrak{b}_-) \mid x \in \mathfrak{g}^r, \mathfrak{b}_+ \in \mathcal{B}_+^x, \mathfrak{b}_- \in \mathcal{B}_-^x\}$$

$$\begin{tikzcd} Z \arrow[d] & (x, \mathfrak{b}_+, \mathfrak{b}_-) \arrow[d, mapsto] \\ X & (x, \mathfrak{b}^-) \end{tikzcd}$$

finite morphism

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\mathfrak{g}}^r \arrow[d] \\ X \arrow[r] & \mathfrak{g}^r \end{tikzcd}$$

$\leftarrow$ finite so finite

$\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$

$(x, \mathfrak{b}) \mapsto x$

$\mathfrak{t} = \mathrm{Lie}\, T$

$\mathfrak{c} = \mathfrak{t}/W = \mathfrak{g}/\!/\mathrm{Ad}\, G$

$Z = \{(x, \mathfrak{b}_+, \mathfrak{b}_-) \mid \}$

$\downarrow$ projection

$\mathcal{B} \times \mathcal{B}$

Actually $Z \to \Omega$

<u>Lemma</u> For any $\mathfrak{b}_+ \in \mathcal{B}_+^x$, $\mathfrak{b}_- \in \mathcal{B}_-^x$, $\mathfrak{b}_+$ and $\mathfrak{b}_-$ are in opposite position, i.e., $\mathfrak{b}_+ \cap \mathfrak{b}_-$ is a cartan.

Ω = G-~~diag~~ diag. orbit in $\mathcal{B} \times \mathcal{B}$ formed by Borels in opposite position

Let $\mathfrak{b}_+, \mathfrak{b}_-$ be any pair of Borels in opposite position.
then

$$\mathcal{O}_+(\mathfrak{b}_+) \xrightarrow{\sim} \mathcal{O}_-(\mathfrak{b}_-)$$

check definitions

$$\mathbb{O}_+(\mathfrak{b}_+) \subset \mathfrak{n}_+/[\mathfrak{n}_+,\mathfrak{n}_+] \xrightarrow{\sim} [\mathfrak{n}_+,\mathfrak{n}_+]^{\perp}/\mathfrak{n}_+^{+} \xrightarrow{\sim} \mathfrak{h}'/\mathfrak{h}$$

← Killing form

$\widetilde{\mathcal{B}}_+ \times \widetilde{\mathcal{B}}_-$

$(\mathfrak{b}_+, x), (\mathfrak{b}_-, x)$

$$\begin{array}{l} Z \\ \downarrow \\ \Omega \subset \mathcal{B}\times\mathcal{B} \xleftarrow{p} \widetilde{\mathcal{B}}_+\times\widetilde{\mathcal{B}}_- \end{array}$$

$p^{-1}(\Omega) \supset \{(\mathfrak{b}_+, x, \mathfrak{b}_-, y) \mid (\mathfrak{b}_+,\mathfrak{b}_-)\in\Omega,\ x\sim y\} ?=: \widetilde{\Omega}$

$\widetilde{\Omega} \subset \widetilde{\mathcal{B}}_+\times\widetilde{\mathcal{B}}_-$ G-diagonal closed orbit

---

$$Y := \tilde{\mathfrak{g}}^r \times_{\mathcal{B}} \widetilde{\mathcal{B}}_+$$

$$\begin{array}{ccc} \tilde{\mathfrak{g}}^r & & \widetilde{\mathcal{B}}_+ \\ & \searrow\ \mathcal{B}\ \swarrow & \end{array}$$

~~$\mathfrak{g}^x \to \mathfrak{g}^x$~~

i.e., $Y = \{(x, \mathfrak{b}_+, y) \mid y \in \mathbb{O}_+(\mathfrak{b}_+),\ x\in\mathfrak{g}^r,\ x\in\mathfrak{b}_+\}$

$$\begin{array}{l} \mathfrak{g}^x \\ \downarrow \qquad T\text{-torsor} \\ \tilde{\mathfrak{g}}^r \end{array}$$

Fix $\underbrace{sl_2}_{\text{principal}}$-triple $(e,h,f)\in\mathfrak{g}$

$f\in\mathfrak{b}$

$X = G\times_{\mathcal{B}}^{U}(e+\mathfrak{b})$

$\downarrow$

$\mathfrak{g}^r$

$\tilde{\mathfrak{g}} = G\overset{B}{\times}\mathfrak{b}$

$\tilde{\mathfrak{g}}^r = G\overset{B}{\times}\mathfrak{b}^r$

$Y = G\overset{U}{\times}\mathfrak{b}^r$

written this way you don't see the W-action (on Y)

$$\begin{array}{ccc} Z = \{(x, \mathfrak{b}_+, \mathfrak{b}_-) \mid \mathfrak{b}_+ \in \mathcal{B}^x_+ , \mathfrak{b}_- \in \mathcal{B}^x_-\} & & (x, \mathfrak{b}_+, \mathfrak{b}_-) \\ \downarrow & & \downarrow \\ \tilde{\Omega} & & (y, \mathfrak{b}_+, \mathfrak{b}_-) \end{array}$$

$$\begin{array}{c} \tilde{\Omega} \subset \tilde{\mathcal{B}}_+ \times \tilde{\mathcal{B}}_- \\ \downarrow \text{ T-torsor} \\ \Omega = \{(\mathfrak{b}_+, \mathfrak{b}_-)\} \subset \mathcal{B} \times \mathcal{B} \end{array}$$

$x \bmod \mathfrak{b}_- \in \mathcal{O}_-(\mathfrak{b}_-)$

$y \in \mathcal{O}_+(\mathfrak{b}_+) \longleftrightarrow x$

$Z \longrightarrow \tilde{\mathfrak{g}}^r \qquad (x, \mathfrak{b}_+, \mathfrak{b}_-) \longmapsto (x, \mathfrak{b}_+)$

$Y = \{(x, \mathfrak{b}_+) \mid y \in \mathcal{O}_+(\mathfrak{b}_+)\}$

(Kostant)

$$\begin{tikzcd} \tilde{\mathfrak{g}}^r \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{g}^r \arrow[d] \\ \mathfrak{t} \arrow[r] & \mathfrak{c} \end{tikzcd} \qquad (x, \mathfrak{b}) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}]$$

$$\tilde{\mathfrak{g}}^r \xrightarrow{\sim} \mathfrak{g}^r \times_{\mathfrak{c}} \mathfrak{t}, \qquad W \circlearrowright \tilde{\mathfrak{g}}^r, \quad W \circlearrowright \mathfrak{g}^r \times_{\mathfrak{c}} \mathfrak{t}$$

<u>Prop</u> $\exists$ W-action on Y s.t. $\underbrace{Y \longrightarrow \tilde{\mathfrak{g}}^r}_{T\text{-torsor}}$ is W-equivariant, compatible with T-action in a 'twisted' way.

$W \curvearrowright Z = \{(x, \mathfrak{b}_+, \mathfrak{b}_-)\}$ induced by W-action on $\tilde{\mathfrak{g}}^r$

<u>Lemma</u> $\xi : Z \longrightarrow Y$

$(x, \mathfrak{b}_+, \mathfrak{b}_-) \longmapsto (x, \mathfrak{b}_+, y)$

W ~~resp~~ action on Z takes any fiber of ξ to a fiber of ξ.

(This proves the Prop).

symplectic viewpoint

$(-,-)$ Killing form

~~...~~ $\psi(-) = (e,-) \in \mathfrak{g}^*$ $\qquad \mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$

$X = G \overset{U}{\times} (\psi + \mathfrak{n}^{\perp}) = T^{\psi}(G/U)$

$Y \subset G \overset{U}{\times} (\mathfrak{n}^{\perp})^{reg} \subset G \overset{U}{\times} \mathfrak{n}^{\perp} = T^*(G/U)$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, dashed] & Y \\ X & X \times Y \end{tikzcd}$$

$Z \hookrightarrow X \times Y = T^*(G/U) \times T^*(G/U)$

$\tilde{\mathcal{B}}_+ = G/U = \tilde{\mathcal{B}}_-$

$\tilde{\Omega} \subset G/U \times G/U$

Prop Z is a smooth, closed Lagrangian subvariety. Moreover, $Z = N^{\psi}_{\tilde{\Omega}}$

Another viewpoint

$$\begin{tikzcd} X \arrow[d, "\mathfrak{g}\text{-torsors}"] & Z \arrow[l] \arrow[r] \arrow[d, "\pi^*\mathfrak{g}"] & Y \arrow[dl, "T\text{-torsor}"] \\ \mathfrak{g}^r & \tilde{\mathfrak{g}}^r \arrow[l, "\pi"] & \end{tikzcd}$$

$\pi^*\mathfrak{g}$ pullback of $\mathfrak{g}$ ~~over~~ via π.

Lemma (Beilinson – Kazhdan) (unpublished)

If $x \in \mathfrak{b} \Rightarrow \mathfrak{g}^x \in \mathfrak{b}$

$x \in \mathfrak{g}^{reg}$

$$\begin{tikzcd} G^x \arrow[r, hook] & B \arrow[d] \\ & B/[B,B] = T \end{tikzcd}$$

(probably need simply connected, so exercise some care)

$\exists$ can: $\pi^*\mathfrak{g} \to T_{\tilde{\mathfrak{g}}^r}$

$$\begin{tikzcd} \pi^*\mathfrak{g} \times Z \arrow[r, "\text{action}"] \arrow[d, "\text{can} \times \xi"] & Z \arrow[d, "\xi"] \\ T_{\tilde{\mathfrak{g}}^r} \times Y \arrow[r, "\text{action}"] & Y \end{tikzcd}$$

use this to prove the lemma on ξ-fibres.

C. Procesi (Follow up on possible connection w/ Ginzburg talk)

G

$\Delta \subset \mathfrak{g} \oplus \mathfrak{g}$

$(G \times G) \cdot \Delta \subseteq Gr(\mathfrak{g} \oplus \mathfrak{g})$

$\overline{G} := \overline{(G \times G) \cdot \Delta}$ } orbit can be thought of as G (look at stabilizer) so $\overline{G}$ is a compactification of G!

$\overline{G}$ - wonderful compactification

1) $\overline{G}$ smooth $\qquad r = rk(G)$

2) $G \subset \overline{G}$; $D_1, \dots, D_r$ divisors

$\overline{G} = G \cup D_1 \cup \dots \cup D_r$

$2^r \quad G \times G$ orbits

$D_1 \cap \dots \cap D_r = \mathcal{B} \times \mathcal{B}$

There is a natural/canonical way to identify the D_i's w/ simple roots.

$G \times G$

$L^1 \oplus \dots \oplus L^r$ } conormal bundles to D_i's ?

$\downarrow$

$\mathcal{B} \times \mathcal{B}$

$(\mathbb{C}^*)^r$ - torsor. This should be the same as the Ginzburg picture ?.

M. Brion — **More followup**

Look at G action on $\bar{G}$ by 'conjugation'

$$G \xrightarrow{\Delta} G \times G$$

$x \in \mathfrak{g}^{r,ss}$: $T = \underset{\text{open}}{G^x} \subset \underset{\text{smooth}}{\bar{G}^x}$

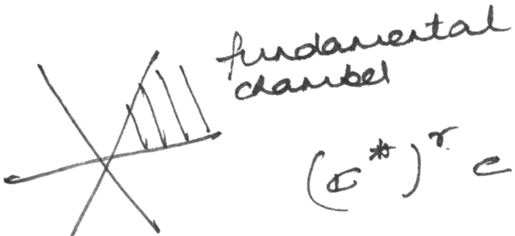
$\bar{G}^x \cap (\mathcal{B} \times \mathcal{B})$

= W-orbit of opposite Borels

suggestive that this corresponds to Ginzburg picture

C. Procesi — **Return after help from M. Brion!**

$x \in \mathfrak{g}^{r,ss}$ $\quad \bar{G}^x$



$(\mathbb{C}^*)^r \subset \mathbb{C}^r$ somehow corresponds to intersection w/ D_i (I didn't understand)

- Bruhat decomposition can be extended to 'Beyinski-Bruhat'-decomposition (by intersection w/ open orbit?)
- By construction $\bar{G}^x$ has a tautological bundle of Lie algebras, that is a 'twisted' cotangent bundle. (could be wrong, Procesi says he thought about this 30 yrs ago, memory haze).
- $D_{i_1} \cap \dots \cap D_{i_k} \to G/P \times G/P^{op}$ $\quad (i_1 - i_k) \subset$ Dynkin diagram

  fibres are the wonderful model of corresponding adjoint group + ... (didn't catch last part)

5 conics in general position, then there are 3264
conics tangent to all 5.

What does this have to do w/ wonderful compactifi-cations?

can do wonderful ... for several varieties, in particular
for conics.

conic picture  wonderful comp..

2-divisors

dual conic

(something I didn't catch)

⇝ the 3264 is now some computation in cohomology.

- maybe interesting to do this for G/B (as a $\mathcal{D}$-affine variety)

do what?

can get full $U(\mathfrak{g})$ as global sections
of $\mathcal{D}_X$ on some X (what X?)

use boundary divisors as some sort of translation functors. This X can't be expected to be $\mathcal{D}$-affine