Taylor Series Expansion

Theorem. *Suppose that $f : U \to \mathbb{C}$ is complex differentiable on a disk $B = B(a, R)$. Then the Taylor series of f at a,*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \frac{f'''(a)}{6}(z-a)^3 + \cdots$$

converges absolutely to f on B, and uniformly on any proper subdisk $B(a, r)$.

Proof. We need only show that f *has* the power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

at every point $z \in B(a, R)$, for this implies that f is analytic at a with a radius of convergence at least R. Thus, the absolute and uniform convergence statements will be consequences of analyticity.

So, fix $z \in B(z, R)$. Choose a radius r such that $|a - z| < r < R$, and consider the curve $\gamma = \{|\zeta - a| = r\}$. Note that z lies on the inside of this circle, and moreover, the Cauchy Integral Formula asserts that

$$f^{(n)}(a) = \frac{n!}{2\pi\, i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta.$$

Now, observe that

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = \sum_{n=0}^{\infty} \frac{1}{2\pi\, i} \int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta\, (z-a)^n = \frac{1}{2\pi\, i} \sum_{n=0}^{\infty} \int_\gamma \frac{f(\zeta)}{\zeta - a}\left(\frac{z-a}{\zeta-a}\right)^n d\zeta.$$

At this point, we'd like to exchange the summation and the integral, which is valid provided the series

$$\zeta \mapsto \sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta - a}\left(\frac{z-a}{\zeta-a}\right)^n$$

converges uniformly on a neighborhood of the curve γ. However, observe that since f is continuous on γ, it is bounded over γ (say by M), whence

$$\left|\left(\frac{f(\zeta)}{\zeta-a}\right)\left(\frac{z-a}{\zeta-a}\right)^n\right| = \left|\frac{f(\zeta)}{\zeta-a}\right|\left|\frac{z-a}{\zeta-a}\right|^n \le \frac{M}{r}\left|\frac{z-a}{r}\right|^n =: M_n.$$

By the choice of r, the expression in modulus is a real number less than 1, whence the series $\sum M_n$ is a convergent geometric series. But then the Weierstrass M-Test asserts the series of functions converges uniformly and absolutely. In fact, using the geometric series again, we see

$$\sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta-a}\left(\frac{z-a}{\zeta-a}\right)^n = \frac{f(\zeta)}{\zeta-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n = \frac{f(\zeta)}{\zeta-a}\,\frac{1}{1-\frac{z-a}{\zeta-a}} = \frac{f(\zeta)}{\zeta-z}.$$

Thus, we can write

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = \frac{1}{2\pi\, i}\int_\gamma \sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta-a}\left(\frac{z-a}{\zeta-a}\right)^n d\zeta = \frac{f(\zeta)}{\zeta-z}\, d\zeta = f(\zeta),$$

where the last equality follows from the Cauchy Integral Formula. □