Uniqueness consequences of Cauchy's Integral Formula

- **Boundary uniqueness.** We stated this yesterday, but this is a good place to repeat it. If f, g are holomorphic on a neighborhood of a Jordan curve γ, and $f(z) = g(z)$ for every $z \in \gamma$, then $f(z) \equiv g(z)$ for every point $z \in \text{inside}(\gamma)$ as well.

- **Cauchy's Inequalities.** Suppose that $f : U \to \mathbb{C}$ is analytic. If $\overline{B(a_0, R)} \subset U$ and $\big|f(z)\big| \le M_R$ on the circle $\gamma = \big\{|z - z_0| = R\big\}$, then
$$\big|f^{(n)}(z_0)\big| \le \frac{n!\, M_R}{R^n}, \qquad n = 1, 2, 3, \ldots.$$

- **Application 1: Liouville's Theorem.** A function is called *entire* if it is $\mathbb{C}$-differentiable at every point in $\mathbb{C}$. *Liouville's Theorem* states that a bounded, entire function is necessarily constant.

- **Application 2: The Fundamental Theorem of Algebra.** A nonconstant complex polynomial has a complex root. In fact, this can be refined to say that a nonconstant complex polynomial of degree d has exactly d complex roots, counting multiplicity.

- **Holomorphic functions have complex values.** Suppose that $f(z)$ is holomorphic and either $f(z)$ is always real or $f(z)$ is always purely imaginary. Then f is a constant.

Root functions

- **Root functions.** The n-th root function $w = z^{1/n}$, where n is a positive integer, is defined as an inverse to the power function $z = w^n$.

- **Multifunctions.** Since z^n wraps the plane over itself n times, there are *exactly* n *different* ways to define a root which differ by a factor of $e^{2\pi\, i/n}$, which is precisely $1/n$-th a full period. Thus, any root function takes the form
$$z^{1/n} = \sqrt[n]{|z|}\, e^{i\, \arg(z)/n} \cdot \big(e^{2\pi i/n}\big)^{k_z}, \qquad k_z \in \mathbb{Z}.$$
If we have a root function $z^{1/n}$ defined on an open set U, then it is called a *branch* of the n-th root if it is continuous on U. In fact, any branch of a root is holomorphic with derivative
$$\frac{d}{dz}\big\{z^{1/n}\big\} = \frac{1}{n}\frac{z^{1/n}}{z}.$$
Observe that the domain U of a branch of a root cannot contain 0, since every neighborhood of 0 has multiple preimages; moreover, a maximal domain consists of a *branch cut* of points (starting at 0) in $\mathbb{C}$ removed from the domain of definition.

- **The principal branch of the root.** If we restrict the argument to the principal argument, then we can define the *principal branch* of the n-th root by
$${z_p}^{1/n} := \sqrt[n]{|z|}\, e^{i\, \arg_p(z)/n}, \qquad -\pi < \arg_p(z) < \pi.$$
Notice that ${z_p}^{1/n}$ is not continuous across its branch cut line $\{x \le 0\}$. A consequence of this is that, in general,
$$(z\, w)^{1/n} \ne z^{1/n}\, w^{1/n}.$$

Harmonic functions

- **The main question.** Let's consider the *Real Part Question:* when is a real-valued function $u(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ the real part of a holomorphic function? Certainly u must be C^∞; however, more can be said. It follows from the CR equations that if $f = u + i\,v$ is holomorphic on U, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial y}\right\}$$
$$= \frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{-\frac{\partial v}{\partial x}\right\} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0,$$

  and similarly for v.

- **Harmonic functions.** A real-valued function $u : U \subset \mathbb{C} \to \mathbb{R}$ is called *harmonic* on U if it is of class C^2 and satisfies *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) \equiv 0 \qquad \forall\, z \in U.$$

  The real and imaginary parts of any holomorphic function are themselves harmonic.

- **Harmonic conjugates.** Let u be a harmonic function on U. A *harmonic conjugate* to v is a harmonic function on U such that $f := u + i\,v$ is holomorphic on U. Equivalently, v is a C^1 function on U that satisfies the CR equations with u. Note that our question becomes: when does a harmonic function have a harmonic conjugate?

- **Uniqueness of harmonic conjugates.** Harmonic conjugates are unique up to a real constant. That is, if $v_1, v_2 : U \to \mathbb{R}$ are two harmonic conjugates to u on U, then $v_2 = v_1 + K$ for some K.

- **Existence of harmonic conjugates.** If U is simply connected and $u : U \to \mathbb{R}$ is harmonic, then a harmonic conjugate v exists on U. Moreover, v can be found by integrating up the CR equations with u. (Specifically, v takes the form

$$v(z) = \int_{\gamma[z_0,z]} -\frac{\partial u}{\partial y}\,dx + \frac{\partial y}{\partial x}\,dy + K,$$

  where K is an arbitrary real constant and $\gamma[z_0, z]$ is any arc in U from an arbitrary fixed point z_0 to z, but I wouldn't recommend memorizing this.)

- **Simply connected is important!** The example $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ is an example of a harmonic function on $\mathbb{C} \setminus \{0\}$ which does *not* possess a harmonic conjugate.