23 Jan 2009 — Algebraic Lie Theory at the Newton Institute

R. Rouquier — Higher Representations of Lie Algebras V

Γ - quiver w/ vertices I, arrows (no loops)
$\rightsquigarrow$ associate cartan matrix

$$a_{ij} = -m_{ij}$$
$$m_{ij} = d_{ij} + d_{ji} \; ; \; d_{ij} = \#\text{arrows } i \to j \quad i \neq j$$
$$a_{ii} = 2$$

$\operatorname{Rep}(\Gamma)$ = stack of (finite dim) representations of Γ over $\mathbb{C}$

$$\text{rep of } \Gamma = \{ \underbrace{L = \{L_i\}_{i \in I}}_{\substack{\text{finite dim} \\ \text{vec. space over } \mathbb{C}}} , \{ f_a : L_i \to L_j \}_{a : i \to j} \}$$

$$\operatorname{Rep}(\Gamma) = \bigsqcup_{\underbrace{\mu \in \oplus \mathbb{Z}_{\geq 0} \alpha_i}_{\text{call this } K^+}} \operatorname{Rep}_\mu(\Gamma) \, ,$$

$$\operatorname{Rep}_\mu(\Gamma) = \{ L \mid \underbrace{\dim L}_{\sum_{i \in I} (\dim L_i) \alpha_i} = \mu \}$$

$$\mathcal{E}_\mu = \{ (L \subset L') , \; L, L' \in \operatorname{Rep}(\Gamma), \; \dim L'/L = \mu \}$$
$(\mu \in K^+)$.

construct geometrically $E_{i_1}^{(n_1)} \cdots E_{i_r}^{(n_r)}$

$$\mathcal{E} = \operatorname{Rep}(\Gamma) \times_{\operatorname{Rep}} \mathcal{E}_{n_r \alpha_{i_r}} \times_{\operatorname{Rep}} \cdots \times_{\operatorname{Rep}} \mathcal{E}_{n_1 \alpha_{i_1}}$$

$$\simeq \{ (0 = L_r \subset L_{r-1} \subset \cdots \subset L_0) \mid \dim L_{m-1}/L_m = n_m \alpha_{i_m} \}$$

$$\begin{array}{ccc} & \downarrow \pi & \mapsto \\ & \operatorname{Rep} & L_0 \end{array}$$

<u>Thm</u> There is an isomorphism of graded $\mathbb{C}$-algebras

$$\mathrm{End}_{\mathcal{B}_{\#\mathbb{C}}}\left(E_{i_1}^{(\alpha_1)} \cdots E_{i_r}^{(\alpha_r)}\right) \xrightarrow{\sim} \bigoplus_{d \geq 0} \mathrm{Ext}^d(\pi_! \mathbb{C}, \pi_! \mathbb{C})$$

(Also a version: $\mathcal{B}_{\#} \xrightarrow{\sim}$ cat. from $\pi_! \mathbb{C}$, various n_i)

<u>construction</u> of <u>the map</u>:

$x_i \in \mathrm{End}(E_1)$? $\mathcal{E}_{\alpha_i \kappa}$

$T_{ij} : E_i E_j \longrightarrow E_j E_i$ $\quad \mathcal{E}_{\alpha_i} \times_{Rep} \mathcal{E}_{\alpha_j}$, $\mathcal{E}_{\alpha_j} \times_{Rep} \mathcal{E}_{\alpha_i}$

$$\mathcal{E}_{\alpha_i} = \{ (L \subset L') \mid \dim L'/L = \alpha_i \}$$

line bundle $(L'/L)_i$; $\underset{\substack{\uparrow \\ \text{chern class}}}{c_1}((L'/L)_i) \in H^2(\mathcal{E}_{\alpha_i})$

$\left(\mathcal{E}_\mu \xrightarrow{\rho_\mu} Rep \times Rep \right.$
$\left. \mathcal{F}_\mu = \rho_{\mu !}(\mathbb{C}_{\mathcal{E}_\mu}) \right)$

$x_i \in \mathrm{Ext}^2_{Rep \times Rep}(\mathcal{F}_{\alpha_i}, \mathcal{F}_{\alpha_i})$

direct img of constant sheaf

<u>$i \neq j$</u>

$$\mathcal{E}_{\alpha_i} \times_{Rep} \mathcal{E}_{\alpha_j} = \{ (L \subset L' \subset L'') \mid \dim L''/L' = \alpha_j, \ \dim L'/L = \alpha_i \}$$

$$Z = \{ (L \subset L'') \mid L''/L \cong \mathbb{C}_{\alpha_i} \oplus \mathbb{C}_{\alpha_j} \} \longrightarrow \{ (L \subset L'') \mid \dim (L''/L) = \alpha_i + \alpha_j \}$$

$$\mathcal{E}_{\alpha_j} \times_{Rep} \mathcal{E}_{\alpha_i} = \{ L \subset L' \subset L'' \mid \dim L''/L' = \alpha_i, \ \dim L'/L = \alpha_j \}.$$

set $\mathcal{F}_{\alpha_i,\alpha_j} = \rho_{(\alpha_i,\alpha_j)!}\mathbb{C}$; $\rho_{\alpha_i,\alpha_j} : \mathcal{E}_{\alpha_i} \times_{Rep} \mathcal{E}_{\alpha_j} \to Rep \times Rep$

$\mathcal{F}_{\alpha_i,\alpha_j} \xrightarrow{T_{ij}} \mathcal{F}_{\alpha_j,\alpha_i}$ [shift]

vector bundle $\mathcal{X} = \mathrm{Ext}^1(L''/L', L'/L)$
on $\mathcal{E}_{\alpha_i,\alpha_j}$ $\quad \| \quad \mathbb{C}^{d_{ji}} \otimes L'/L \otimes (L''/L')^*$

canonical section:
$(L \subset L' \subset L'') \longmapsto$ class of extension
$0 \to L'/L \to L''/L \to L''/L' \to 0$

zeroes $= Z$

$$T_{ij} T_{ji} \overset{?}{=} \varphi_{ij}(x_i \varepsilon_j, \varepsilon_i x_j)$$
$$\| $$
$$(-1)^{d_{ij}} (x_i - x_j)^{m_{ij}}$$

by self-intersection formula

T_{ii} ? $\quad \mathcal{E}_{\alpha_i,\alpha_i} = \{ (L \subset L' \subset L'') \mid \dim L''/L' = \dim L'/L = \alpha_i \}$

$\downarrow$ $\mathbb{P}^1$-bundle assoc. w/ vec. bundle L''/L

$\mathcal{E}_{2\alpha_i} = \{ (L \subset L'') \mid \dim L''/L = 2\alpha_i \}$.

$$T_{ii} : Rp_* \mathbb{C} \xrightarrow[\text{trace}]{} \mathbb{C}[-2] \xrightarrow{\sim} p_* \mathbb{C}[-2] \to Rp_* \mathbb{C}[-2].$$

Relations involving x_i, T_i are classical

Left to check the braid relations

when the braid relations do hold: classical

<u>Note</u>: the missing relation $R=0$.

The other relations give:

$(x_i - x_{i+2})R = 0$, (but Hecke algebras are free over $\mathbb{C}[x_1, \ldots, x_n]$.)

Enough to show that

$\mathrm{Ext}^*(\pi_! \mathbb{C}, \pi_! \mathbb{C})$ free over $\mathbb{C}[x_1, \ldots, x_n]$ or over $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$

= equiv. cohomology of a pt (same as classical).

↑

see Chriss and Ginzburg.

To finish the proof: alg. are modules of finite rank over $\mathbb{C}[x_1, \ldots, x_n]$.

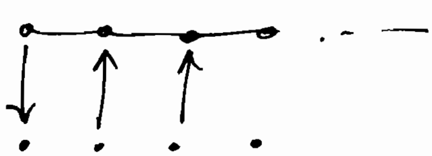
<u>Cor</u> Basis of $K_0(\mathcal{B}_{\mathbb{C}}^{idem})$ given by indecomposable objects is Lusztig's canonical basis of $U(\mathfrak{n}^+)$ (also quantum version)

(Hao-Zheng): construction of $L(\lambda)$, $\lambda \in X^+$ as

$$K_0(D^b(\mathrm{Rep}_{\mu'}(\Gamma'))/(n)) := T$$

Γ = •—•—•—• ⋯ $\circlearrowleft_{\mathbb{Z}/4}$

Γ' =



<u>Thm</u>

$\mathcal{L}(\lambda) \simeq$ full subcategory of T w/ direct sums of ~~simple~~ shifted simple perverse sheaves.

↑

of 2-reps

<u>cor</u> canonical basis of $\mathbf{L}(\lambda)_{\mathbb{A}}$ = basis of indecomposable objects of $\mathcal{L}(\lambda)_{\oplus}^{\mathrm{idem}}$