16 Jan 2009   Algebraic Lie Theory at the Newton Institute

A. Ram   Symmetry, polynomials and quantization IV

symmetric functions

$$\left.\begin{array}{l}\mathfrak{h}_{\mathbb{Z}}\\ \mathfrak{h}^*_{\mathbb{Z}}\end{array}\right\}\ \text{dual } \mathbb{Z}\text{-vector spaces}$$

$$\langle\cdot,\cdot\rangle : \mathfrak{h}^*_{\mathbb{Z}} \times \mathfrak{h}_{\mathbb{Z}} \longrightarrow \tfrac{1}{e}\mathbb{Z}$$

W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections.

W_0 acts on the group algebra of $\mathfrak{h}_{\mathbb{Z}}$:

$K_{T^\vee}(pt) = \text{span}\{Y^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$ w/ $Y^{\lambda^\vee}Y^{\sigma^\vee} = Y^{\lambda^\vee+\sigma^\vee}$ and $wY^{\lambda^\vee} = Y^{w\lambda^\vee}$

The algebra of symmetric functions

$K_{T^\vee}(pt)^{W_0} = \{f \in K_{T^\vee}(pt) \mid wf = f \text{ for all } w \in W_0\}$

Then $K_{T^\vee}(pt)^{\det} = \{f \in K_{T^\vee}(pt) \mid wf = \det(w) f \text{ for all } w \in W_0\}$ is a free $K_{T^\vee}(pt)^{W_0}$-module of rank 1.

$K_{T^\vee}(pt)^{W_0}$ and $K_{T^\vee}(pt)^{\det}$ have bases

$$m_{\lambda^\vee} = \Pi_0 Y^{\lambda^\vee} \quad \text{and} \quad a_{\lambda^\vee+\rho^\vee} = \varepsilon_0 Y^{\lambda^\vee+\rho^\vee}$$

where $\Pi_0 = \sum_{w\in W_0} w$ and $\varepsilon_0 = \sum_{w\in W_0} \det(w^{-1})\, w$

and $\lambda^\vee \in P^+ = \mathfrak{h}_{\mathbb{Z}}/W_0$

$$K_{T^\vee}(pt)^{W_0} \xrightarrow{\sim} K_{T^\vee}(pt)^{\det}$$

naive basis	$m_{\lambda^\vee}$		
	$s_{\lambda^\vee}$	$\longleftrightarrow$	$a_{\lambda^\vee+\rho^\vee}$ naive basis
	f	$\longmapsto$	$a_{\rho^\vee} f$

$m_\lambda^\vee$ are the <u>monomial symmetric functions</u>
$s_\lambda^\vee$ are the <u>Weyl characters</u> or <u>Schur functions</u>
$a_{\rho^\vee}$ is the <u>Weyl denominator</u> or <u>Vandermonde</u>

$$K_{T^\vee}(pt.)^{W_0} = K_{G^\vee}(pt) = K_0(G^\vee\text{-modules})$$

$s_{\lambda^\vee}$ are the class of the simple $G^\vee$-modules.

<u>Double affine Hecke algebra</u> $\tilde{H}$

$\tilde{H}$ has basis $\{ q^{k/e} X^\mu T_w Y^{\lambda^\vee} \mid {k \in \mathbb{Z}, w \in W_0 \atop \mu \in \mathfrak{h}^*_{\mathbb{Z}}, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}} \}$

$\cup$|

$H^\vee = \mathrm{span}\{ X^\mu T_w \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0 \}$

$\cup$|

$H_0 = \mathrm{span}\{ T_w \mid w \in W_0 \}$ w/

$q^{1/e} \in Z(\tilde{H})$, $X^\mu X^\nu = X^{\mu+\nu}$, $Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}$

H_0 is generated by $T_1, \dots, T_n$ and

$$T_i^2 = (t^{1/2} - t^{-1/2}) T_i + 1$$

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}} \qquad \text{and } \frac{\pi}{m_{ij}} = \sphericalangle(\mathfrak{h}^{\alpha_i}, \mathfrak{h}^{\alpha_j})$$

T_i has eigenvalues $t^{1/2}, -t^{-1/2}$

$H^\vee$ has a (unique) 1-dim module
$\mathrm{span}\{\mathbb{1}\}$ w/ $T_i \mathbb{1} = t^{1/2}\mathbb{1}$
The <u>polynomial representation</u> of $\tilde{H}$

$$\mathrm{Ind}_{H^\vee}^{\tilde{H}} \mathbb{1} = \tilde{H}\mathbb{1} = \mathrm{span}\{ q^{k/e} Y^{\lambda^\vee}\mathbb{1} \mid {k \in \mathbb{Z} \atop \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}} \}$$
$$= K_{T^\vee}(pt)\,\mathbb{1}.$$

Define $\mathbb{1}_0, \varepsilon_0$ in H_0 by

$$\mathbb{1}_0 T_i = t^{1/2}\mathbb{1}_0 \quad \text{and } \varepsilon_0 T_i = (-t^{-1/2})\varepsilon_0$$

Then
$$\begin{array}{rcl} K_{T^\vee}(pt) = H\mathbb{1} & \longrightarrow & \mathbb{1}_0 \tilde{H}\mathbb{1} = K_{T^\vee}(pt)^{W_0}\mathbb{1} \\ f\mathbb{1} & \longmapsto & \mathbb{1}_0 f \mathbb{1} \end{array}$$

The <u>non symmetric Macdonald polynomials</u>
$E_{\lambda^\vee} = E_{\lambda^\vee}(q,t)$ in $K_{T^\vee}(pt)$ are given by:

a) E_{λ^r} is an eigenvector for all X^μ
b) $E_{\lambda^r} = Y^{\lambda^r} +$ lower stuff

The symmetric Macdonald polynomials
$P_{\lambda^r} = P_{\lambda^r}(q,t)$ in $K_{T^r}(pt)^{W_0}$ are given by
$$P_{\lambda^r}\mathbb{1} = \mathbb{1}_0 E_{\lambda^r}\mathbb{1}, \quad \lambda^r \in P^+$$
Define $A_{\lambda^r+\rho^r} = A_{\lambda^r+\rho^r}(q,t)$ in $K_{T^r}(pt)$ by
$$A_{\lambda^r+\rho^r}\mathbb{1} = \varepsilon_0 E_{\lambda^r+\rho^r}\mathbb{1}$$

$\Rightarrow$ Warning $\quad \varepsilon_0 \tilde{H}\mathbb{1} \neq K_T(pt)^{\det}\mathbb{1}$

$$K_{T^r}(pt)^{W_0}\mathbb{1} = \mathbb{1}_0\tilde{H}\mathbb{1} \xrightarrow{\sim} \varepsilon_0\tilde{H}\mathbb{1}$$
$$f\mathbb{1} \longmapsto A_{\rho^r}(q,t)\, f\mathbb{1}$$
naive basis $\mathbb{1}_0 E_{\lambda^r}\mathbb{1} = P_\lambda(q,t)\mathbb{1}$

$$P_{\lambda^r}(q,qt)\mathbb{1} \longleftarrow \underbrace{A_{\lambda^r+\rho^r}(q,t)\mathbb{1} = \varepsilon_0 E_{\lambda^r+\rho^r}\mathbb{1}}_{\text{naive basis}}$$

$$K(P_K(G/K))$$
$$\|$$

At $q=0$
$$Z(H)\mathbb{1} = \cancel{Z(H)\mathbb{1}} \simeq K_{T^r}(pt)^{W_0}\mathbb{1} = \mathbb{1}_0\hat{H}\mathbb{1} \xrightarrow{\sim} \varepsilon_0 H\mathbb{1}$$
$$f\mathbb{1} \longmapsto A_{\rho^r}(0,t)\, f\mathbb{1}$$
$$\mathbb{1}_0 Y^{\lambda^r}\mathbb{1} = P_{\lambda^r}(0,t)\mathbb{1}$$
$$s_{\lambda^r}\mathbb{1}_0 = P_{\lambda^r}(0,0)\mathbb{1} \longleftarrow A_{\lambda^r+\rho^r}(0,t)\mathbb{1} = \varepsilon_0 Y^{\lambda^r+\rho^r}\mathbb{1}_0$$

- Remove $\sim$s ; - change $\mathbb{1} \to \mathbb{1}_0$

$P_{\lambda^r}(0,t)$ is the Macdonald spherical function or Hall - Littlewood polynomial
(Macdonald in 1971)

At $q=0$, $t=1$ the above picture becomes the Weyl character formula story.

<u>Remarks</u>: 1) At $q \neq 0$, $Z(\tilde{H})$ is trivial,
i.e $Z(\tilde{H}) = \mathbb{C}[q^{\pm 1/2}]$
At $q=0$, $Z(\tilde{H})$ is big and contains
$Z(H) = K_{T^r}(pt)^{W_0}$ (Thm. of Bernstein)

2) The map:
$$K_{T}(pt)^{W_0}\mathbb{1}_0 \xleftarrow{\sim} \mathbb{1}_0 H \mathbb{1}_0$$
$$P_{\lambda^r}(0,t)\mathbb{1}_0 \qquad \mathbb{1}_0 Y^{\lambda^r}\mathbb{1}_0$$
is the <u>Satake</u> <u>isomorphism</u>

3) H is the Grothendieck ring (under convolution) of I-equivariant perverse sheaves on $\mathcal{G}/I$ (= affine flag variety)

$\mathbb{1}_0 \tilde{H} \mathbb{1}_0$ is the <u>spherical</u> <u>Hecke</u> <u>algebra</u> is the Grothendieck ring of K-equivariant perverse sheaves on $\mathcal{G}/K$ (= the <u>loop</u> <u>grassmanian</u>)

$\{S_{\lambda^r}\mathbb{1}_0\}$ is the Kazhdan-Lusztig basis
$\mathbb{1}_0 \tilde{H} \mathbb{1}_0$

$S_{\lambda^r}\mathbb{1}$ is the image $IC(K h_{\lambda^r}(t^{-1})K, \overline{\mathbb{Q}}_\ell)$