Math 748 Homework 10

Due Wednesday, November 15

1. (a) Show that if α is a root of a monic polynomial $f \in \mathbb{Z}[x]$ and if $f(r) = \pm 1$ for some $r \in \mathbb{Z}$, then $\alpha - r$ is a unit in $\mathbb{Z}[\alpha]$. (Hint: What is the constant coefficient of $g(x) = f(x - r)$?)

   (b) Find the fundamental unit in $K = \mathbb{Q}(\sqrt[3]{7})$ (Hint: $\Delta_K = -1323$.)

   (c) Let α be a root of $x^3 + x - 3$. Find the fundamental unit of the ring of integers of $K = \mathbb{Q}(\alpha)$. (Feel free to use discriminant formulas for cubics.)

2. **The Battle of Hastings** (October 14, 1066; Normans vs. Saxons)

   "The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof, and woe to the hardy Norman who ventured to enter their redoubts; for a single blow of a Saxon war hatched would break his lance and cut his coat of mail... When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle-cries 'Ut', 'Olicrosee!', Godemite!'." [Fictitious historical text, taken from Neukirch (p. 44), who in turn took it from a 1907 book.]

   Question: How many troops did Harold II have at the battle of Hastings?

   Notes:

   (a) Although the above text may be appreciated for its color, here is a clearer restatement: before Harold joins his men, they are in thirteen squares, each square consisting of an equal number of men. Once Harold joins them, they all together rearrange themselves to form a single square.

   (b) Since the world population in 1066 was less than it is today (about 6.7×10^9), there is a unique solution to this problem.

   (c) Your answer should involve finding the fundamental unit in some number field. Continued fractions will work, but in this case you should look for solutions to $a^2 - db^2 = \pm 4$ (think about why this is). You may also prefer trial and error plus an obvious lower bound on ϵ.

   (d) You may use a computer, but only to perform basic arithmetic (+,-,*,/) in $\mathbb{Z}$.

3. Fix an m and M. Is it necessarily true that the set of algebraic integers α in $\mathbb{C}$ of degree $< m$ and with $|\alpha| < M$ is finite? Either prove or give a counterexample. (Recall that in the proof of the Unit Theorem we used a similar fact with the additional hypothesis that every conjugate of α has absolute value $< M$. This allowed us to show that $L(U)$ is discrete.)