MOTIVIC LERAY-HIRSCH

RAHBAR VIRK

Write $DM(k;\Lambda)$ for the triangulated category of motives, over a field k, with Λ-coefficients [CD, Definition 11.1.1]. Write 'scheme' in lieu of 'separated scheme of finite type over k'. There is a covariant functor $X \mapsto M^c(X)$ from the category of schemes and proper morphisms to $DM(k;\Lambda)$ (in the notation of [CD], $M^c(X) = a_* a^! \Lambda$, where $a\colon X \to \mathrm{Spec}(k)$ is the structure morphism). The functor $M^c(X)$ behaves like a Borel-Moore homology theory. The category $DM(k;\Lambda)$ is a symmetric monoidal triangulated category, and $M^c(X \times Y) = M^c(X) \otimes M^c(Y)$. The motive $M^c(\mathrm{Spec}(k))$ is the unit object. It is convenient to set $\Lambda = M^c(\mathrm{Spec}(k))$. Let $H^i_M(X;\Lambda(j))$ denote the motivic cohomology groups of X, as defined in [CD, §11.2]. These are contravariant functors from the category of schemes to Λ-modules. Each $e \in H^i(X;\Lambda(j))$ determines a canonical map

$$e\cap\colon M^c(X)(-j)[-i] \to M^c(X).$$

Proposition 0.1 (Leray-Hirsch)**.** *Let $p\colon X \to Y$ be a morphism of schemes. Assume that Y may be covered by open subschemes U_i such that:*

(i) *for each U_i, there is a finite étale morphism $f_i\colon V_i \to U_i$, with degree invertible in Λ, such that $X \times_Y V_i \simeq V_i \times F$, for some fixed scheme F;*
(ii) *$M^c(F)$ is a direct sum of Tate motives;*
(iii) *there exist $e_1, \ldots, e_n \in H^{2*}(X;\Lambda(*))$ that restrict to a basis of $H^{2*}(F;\Lambda(*))$ for each fibre inclusion $F \to X$.*

Then there is an isomorphism

$$M^c(X) \simeq M^c(Y) \otimes M^c(F).$$

Proof. Let d_i denote the degree of e_i. Set $d = \dim(F)$. Each e_i determines a map

$$M^c(Y)(d-d_i)[2(d-d_i)] \xrightarrow{p^*} M^c(X)(-d_i)[-2d_i] \xrightarrow{e_i\cap} M^c(X).$$

Summing these, we obtain a map

$$\bigoplus_i M^c(Y)(d-d_i)[2(d-d_i)] \to M^c(X).$$

As F is a direct sum of Tate motives, this may be rewritten as a map

$$\theta_Y\colon M^c(Y) \otimes M^c(F) \to M^c(X).$$

The map θ_Y is clearly an isomorphism if p is the projection $Y \times F \to Y$. Let $f_i\colon V_i \to U_i$ be as in the Proposition. It suffices to prove each θ_{U_i} is an isomorphism. As $f_{i*} f_i^*$ is multiplication by the degree of f_i, the result follows. □

References

[CD] D-C. Cisinski, F. Déglise, *Triangulated categories of mixed motives*, arXiv:0912.2110v3.

The Appalachians