Some commentary around 'integration along the fiber' follows. I originally wanted to see, transparently, the compatibility of classical integration along the fibre, Thom isomorphism and some 'wrong way' maps from topology with the usual additional structures in complex geometry (mixed Hodge structures, etc.). This is easily done, but the situation is a bit more interesting than just that. I don't have any particular applications in mind, just leisurely strolling (stumbling) through a forest.

Regardless, here is the précis: *the trace morphism, à la SGA 4 Exposé XVIII, is integration along the fiber*.

1. **Classical constructions.** We have the following classical construction in topology in the absolute (over a point) setting with constant coefficients. Let $\pi\colon E \to B$ be a (Serre) fibration of reasonable spaces (homotopy type of CW-complexes say, not necessarily finite dimensional). Assume the fiber F is finite dimensional of dimension d. In particular, $H^n(F) = 0$ for $n > d$. Define:

$$\pi_\sharp\colon H^*(E) \to H^{*-d}(B; H^d(F))$$

in terms of the Leray-Serre spectral sequence by the composition:

$$H^i(E) \twoheadrightarrow E_\infty^{i-d,d} \hookrightarrow E_2^{i-d,d} = H^{i-d}(B; H^d(F)),$$

where the middle map is the edge map.

By the standard functoriality of the spectral sequence, $\pi_\sharp$ is functorial for morphisms of fibrations (in particular, for base change), and is a map of $H^*(B)$-modules (projection formula):

$$\pi_\sharp(\pi^*(\alpha) \cdot \omega) = \alpha \cdot \pi_\sharp(\omega).$$

It is also functorial for compositions (however, in this setup this is not quite as transparent as I would like).

If M is a compact smooth manifold and cohomology is with $\mathbf{R}$-coefficients, then $\pi_\sharp$ for $\pi\colon M \to *$ coincides with integration at the level of differential forms (modulo orientation). A formal consequence of this and functoriality is that $\pi_\sharp$ coincides with integration along the fiber for smooth bundles of manifolds with compact fiber. If the base is compact this is immediate from Mayer-Vietoris, functoriality for base change and the maps agreeing fiber by fiber.

A variant of this allows us to incorporate a bit more of the structure of the fiber. Suppose $\pi\colon E \to B$ is a fiber bundle of manifolds with compact fiber (of dimension d). For simplicity, assume everything is compatibly oriented. Let $e(T_{E/B})$ be the Euler class of the relative tangent bundle. In particular, $e(T_{E/B})$ restricts to the Euler class of a fiber. Define a modified 'transfer':

$$\pi_\tau \colon H^*(E) \to H^*(B) \quad \text{by} \quad \alpha \mapsto \pi_\sharp(\alpha \cdot e(T_{E/B})).$$

Then

$$\pi_\tau \pi^* = \chi(F) \cdot \mathrm{id},$$

where $\chi(F)$ is the Euler characteristic of the fiber F.

In the situation of manifolds, an equivalent way to accomplish all of this would be to pass from cohomology to Borel-Moore homology, push forward along a proper map, dualize back. However, this transfer exists in much greater generality. It is a special case of what the topologists call Becker-Gottlieb transfer. It exists whenever the base and fiber are homotopy finite. However, the 'textbook' construction goes through spectra and some homotopy theory. This is quite satisfactory for homotopy theory, but not so for those of us eyeing the relative case and sheaf coefficients.

2. **Sheaves.** In the relative setup, i.e., for sheaves, integration along the fiber *should* be a morphism of functors of the form

$$\pi_* \pi^* \to \mathrm{id}[-2d](-d),$$

for nice maps like fibrations say. I have not thought much about this. Instead, I will discuss the 'compact vertical cohomology' variant of this which is closer to classical 'integration along the fiber'. Namely, look for a morphism of functors of the form

$$\pi_! \pi^* \to \mathrm{id}[-2d](-d).$$

As usual, all functors are derived, $2d$ is fiber dimension, $(-d)$ is Tate twist (depending on the context). Perhaps it is better to formulate everything in terms of the 'orientation sheaf', but so that I don't get too confused let me stick to using Tate twists for now.

There is more or less nothing that needs to be done if we restrict ourselves to (oriented) topological submersions: the usual adjunction gives us the desired maps. However, as the absolute situation indicates, we should be able to do better (we should certainly be able to make it work for fiber bundles). Regardless, any general construction should coincide with the the special case of topological submersions and satisfy all the functoriality one could possibly ask for (compatibility with base change, etc.). The functoriality should characterize said map uniquely (when restricted to interesting categories of spaces).

With this perspective it is quite easy to address my initial motivation (compatibility with mixed structures in complex algebraic geometry), let me digress briefly and get this out of the way.

3. **Compact vertical cohomology, Thom isomorphism and mixed structures.** I will play with varieties (separated schemes of finite type over $\mathrm{Spec}(\mathbf{C})$). The 'constant sheaf' will be denoted by $\underline{X}$. Forgive me the sin of not strictly defining what category of 'sheaves' I am working with: at this point I will only use a very limited version of the usual yoga - localization triangle, !-pushforward, !-pullback only for closed immersions. So ordinary sheaves (Noetherian ring coefficients), mixed Hodge modules, motivic sheaves, ... all work. The term 'sheaf' should be interpreted as an arbitrary object in the derived/triangulated category one chooses to work with.

Let $\pi\colon E \to B$ be a vector bundle, of (complex) rank d with zero section $i\colon B \hookrightarrow E$. Apply $\pi_!$ and π_* to the canonical maps $i_*i^! \to \mathrm{id}$ and $\mathrm{id} \to i_*i^*$. Then we have our old friend, the algebraic homotopy lemma:

$$i^!\underline{E} \xrightarrow{\sim} \pi_!\underline{E} \quad \text{and} \quad \pi_*\underline{E} \xrightarrow{\sim} i^*\underline{E}$$

are isomorphisms. The composition:

$$\pi_!\pi^*\underline{B} \xrightarrow{\sim} \pi_!\pi^!\underline{B}[-2d](-d) \xrightarrow{\text{adjunction}} \underline{B}[-2d](-d) \tag{3.1}$$

is clearly an isomorphism. Combined with the homotopy lemma we get:

Proposition 3.2 (Thom isomorphism). *Let $\pi\colon E \to B$ be a vector bundle of rank d. Write $i\colon B \hookrightarrow E$ for the zero section. Then we have a canonical isomorphism:*

$$i^!\underline{E} \xrightarrow{\sim} \pi_!\underline{E} \xrightarrow{(3.1)} \underline{B}[-2d](-d).$$

Push everything to a point to get the 'classical Thom isomorphism':

$$H^*_B(E) \xrightarrow{\sim} H^*_{cv}(E) \xrightarrow{\sim} H^{*-2d}(B)(-d).$$

(I hope the notation is self-evident). The compatibility with any mixed structures around is now transparent.

4. **Trace maps.** Back to seeking a map of the form

$$\pi_\sharp\colon \pi_!\pi^* \to \mathrm{id}[-2d](-d).$$

There is a relatively easy answer for varieties. We can do this for *any* morphism of varieties!

What makes everything tick is already manifest in the standard construction of fundamental classes for *arbitrary* varieties. Namely, for an irreducible variety X of (complex) dimension d, let $U \subset X$ be the smooth locus. Then U is dense in X and $X - U$ has *strictly* lower dimension than X. So we have an isomorphism:

$$H_c^{2d}(U) \xrightarrow{\sim} H_c^{2d}(X).$$

Thus, a fundamental class of U yields a fundamental class for X. The point: generic smoothness (of X), the fact that $\mathbf{C}$ has real dimension 2, and cohomological dimension (for compact supports) is bounded by the (real) dimension, allows us to reduce everything in the top dimension to smooth varieties.

The relative version of this is not much more complicated. At this point I will restrict myself to working with ordinary sheaves or mixed Hodge modules. The arguments are greatly simplified by the presence of the standard (non-perverse) t-structure. Without a t-structure (motivic sheaves) the arguments require a lot more consideration of distinguished triangles and long exact sequences of Hom-groups (Chow groups in the motivic setting). It would just obsfuscate the core simplicity of the situation.

It will be convenient to make the following definition: for a morphism of varieties $\pi\colon E \to B$, set

$$d_\pi = \text{maximum of the dimensions of (geometric) fibers of } \pi.$$

We want to construct a canonical map

$$\pi_\sharp\colon \pi_! \pi^* \to \mathrm{id}[-2d_\pi](-d_\pi).$$

By the projection formula I only need to do this for the constant sheaf. Further, we may assume that all our varieties are reduced.

Now the point is that the cohomology sheaves of $\pi_! \underline{E}$ vanish above degree $2d_\pi$. So the sought after map determines, and is uniquely determined by, a map:

$$H^{2d_\pi}(\pi_! \underline{E}) \to \underline{B}(-d_\pi).$$

If $\pi\colon E \to B$ is a morphism of *irreducible* varieties, then there is an open dense subvariety $U \subset E$ such that the restriction of π to U, denoted π_U, is smooth. As in the absolute case, for dimension reasons, the canonical map $\pi_{U!} \underline{U} \to \pi_! \underline{E}$ yields an isomorphism:

$$H^{2d_\pi}(\pi_{U!} \underline{U}) \xrightarrow{\sim} H^{2d_\pi}(\pi_! \underline{E}).$$

Hence,

$$\pi_{U!}\pi^* \xrightarrow{\sim} \pi_{U!}\pi_U^![-2d_\pi](-d_\pi) \xrightarrow{\text{adjunction}} \mathrm{id}[-2d_\pi](-d_\pi)$$

yields the desired map $\pi_\sharp$.

To deal with multiple components, glue the $\pi_\sharp$ obtained above for each component (they agree on intersections).

Theorem 4.1. *To each morphism of varieties $\pi\colon E \to B$, we may associate, in a unique way, a canonical map, the* trace map*:*

$$\pi_\sharp\colon \pi_!\pi^* \to \mathrm{id}[-2d_\pi](-d_\pi),$$

such that the following conditions are satisfied:

(i) *If*

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ \tilde{B} \arrow[r, "f"'] & B \end{tikzcd}$$

is a cartesian square, then the diagram

$$\begin{tikzcd} \tilde{\pi}_!\tilde{\pi}^*f^* \arrow[r, "\sim"] \arrow[dr, "\tilde{\pi}_\sharp"'] & \tilde{\pi}_!\tilde{f}^*\pi^* & f^*\pi_!\pi^* \arrow[l, "\sim"'] \arrow[dl, "f^*(\pi_\sharp)"] \\ & f^*[-2d](-d) & \end{tikzcd}$$

where the horizontal arrow on the right is the base change isomorphism, and $d = d_\pi = d_{\tilde{\pi}}$, is commutative.

(ii) *Given a sequence of morphisms $E_1 \xrightarrow{g} E_2 \xrightarrow{f} B$, the diagram*

$$\begin{tikzcd} f_!g_!g^*f^* \arrow[r, "f_!(g_\sharp)"] \arrow[d, "\sim"'] & f_!f^*[-2d_g](-d_g) \arrow[d, "f_\sharp"] \\ (fg)_!(fg)^* \arrow[r, "(fg)_\sharp"] & \mathrm{id}[-2(d_g+d_f)](-(d_g+d_f)) \end{tikzcd}$$

is commutative.

(iii) *If π is smooth of relative dimension d, then $\pi_\sharp$ is the composition:*

$$\pi_!\pi^* \xrightarrow{\sim} \pi_!\pi^![-2d](-d) \xrightarrow{\text{adjunction}} \mathrm{id}[-2d](-d).$$

The existence of $\pi_\sharp$ was sketched above. The uniqueness is clear from the functoriality requirements. Checking these requirements doesn't require any new ideas but involves the usual tedium of ensuring the 'coherence' of base change isomorphisms and adjunctions with compositions, etc.

In the étale cohomology context this result is contained in SGA 4 Exposé XVIII §2. There the goal is to construct the duality isomorphism

$$f^* \xrightarrow{\sim} f^![-2d](-d)$$

for smooth morphisms. The argument in SGA 4 proceeds via a long induction through the method of fibering by curves. It offers an alternate proof of existence. In our complex algebraic/topological situation, the argument above is a large simplification since we already have the duality adjunction/isomorphism for smooth morphisms quite cheaply.

There are now a number of games one can play with characteristic classes (or more generally with interesting classes on the total space whose restrictions to the fibers can be understood) to get relative versions of the transfer as in the absolute setting. I haven't thought about this enough to see if one gets anything interesting - it is probably only interesting in the torsion coefficients setting since over $\mathbf{Q}$ everything is probably subsumed by Hard Lefschetz. There is also the question of compatibility with specialization/cospecialization (á la nearby cycles) that I need to think about a bit.