EQUIVARIANT NEARBY CYCLES

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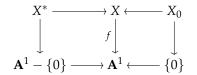
1. **Preliminaries.** We play with complex varieties. Let *G* be an algebraic group and *X* a *G*-variety. A diagram

$$X \xleftarrow{p} X_G \xrightarrow{q} \bar{X}$$

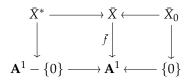
with *p* equivariant and *q* a *G*-torsor will be called a *resolution* of *X*. If *p* is smooth (resp. *n*-acyclic) we will call it a smooth (resp. *n*-acyclic) resolution. We will say that $\overline{M} \in D\overline{X}$ comes from *X* if there exists some $M \in DX$ along with an isomorphism

$$p^*M \simeq q^*\bar{M}$$

Let $f: G \to \mathbf{A}^1$ be a *G-invariant* morphism. That is, $f(g \cdot x) = f(x)$ for all $g \in G$ and $x \in X$. Then we have a diagram



with all squares cartesian (the lower horizontal maps being the evident inclusions). The morphism f induces a morphism $\bar{f} \colon \bar{X} \to \mathbf{A}^1$. So we obtain a diagram



with all squares cartesian. This yields the nearby cycles functor

$$\psi_{\bar{f}} \colon DX \to DX_0.$$

Lemma 1.1. Assume $\overline{M} \in D\overline{X}$ comes from DX. Then $\psi_{\overline{t}}(\overline{M})$ comes from DX_0 .

Proof. By assumption, there exists $M \in DX$ along with an isomorphism

$$p^*M \simeq q^*\bar{M}.$$

As taking nearby cycles commutes with pulling back along smooth morphisms,

$$p^*\psi_f(M) \simeq \psi_{f \circ p}(p^*M) \simeq \psi_{f \circ p}(q^*\bar{M}) \simeq q^*\psi_f(\bar{M}).$$

Theorem 1.2. Assume that X admits n-acyclic smooth resolutions for each n. Then the nearby cycles functor $DX \rightarrow DX_0$ lifts to the equivariant setting. That is, we have a functor, the equivariant nearby cycles,

$$\psi_f: D_G X \to D_G X_0$$

compatible with nearby cycles on DX which satisfies all the usual properties of the ordinary nearby cycles.

Proof. Let $M \in D_G X$. The previous Lemma yields an object $\psi_{\bar{f}} \bar{M} \in D\bar{X}_0$ for each resolution $X_0 \leftarrow E \rightarrow \bar{X}_0$ that comes from X. These objects are compatible under smooth pullback, since the nearby cycles functor is. As X admits *n*-acyclic resolutions, this prescription extends to yield a functor

$$\psi_f: D_G X \to D_G X_0$$

It is clear that this functor has all the required properties.

Proposition 1.3. Let $H \subseteq G$ be a closed subgroup. Then ψ_f commutes with the restriction functor res_H^G . Moreover, if G/H is complete, then ψ_f also commutes with induction from H to G.

Proof. Restriction is given by pulling back along a smooth morphism. So commutation of ψ_f with res^{*G*}_{*H*} is immediate. Similarly, induction is given by a push forward (*- or !- depending on whether we want a right or left adjoint¹ to res^{*G*}_{*H*}). If G/H is complete, then this push forward occurs along a proper morphism. This yields the desired result, since nearby cycles commute with push forwards along proper morphisms.

References

[BL] J. BERNSTEIN, V. LUNTS, Equivariant sheaves and functors, Lecture Notes in Math. 1578, Springer-Verlag, Berlin (1994).

¹ In the case of the !-pushforward one should also incorporate a shift in order to get a left adjoint to res^H_G. However, this is irrelevant to the discussion at hand.